Construction and Analysis of Greedy Sequences of Non-Negative Integers

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Greedy Sequences

Definition of a greedy sequence of non-negative integers:

- A greedy sequence is a strictly increasing sequence of integer numbers, G, with a general term a_n.

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Definition of a greedy sequence of non-negative integers:

- A greedy sequence is a strictly increasing sequence of integer numbers, G, with a general term a_n.
- The sequence begins with $a_0 = 0$.
- Every next member a_j is defined as the least natural number such that $a_j > a_{j-1}$, the condition $a_{i_1} + a_{i_2} + a_{i_3} + ... + a_{i_y} \neq p \cdot a_{i z}$, is true for all $i_1, i_2, ..., i_z \in [0; j]$ where p is a fixed real number.

Example (Stanley sequence)

The greedy sequence that avoids forming an arithmetic progression:

- Equation to be met: $a_a + a_b \neq 2 \cdot a_c$.

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The greedy sequence that avoids forming an arithmetic progression:

- Equation to be met: $a_a + a_b \neq 2 \cdot a_c$.
- Initially defined terms: $a_0 = 0, a_1 = 1$.
- Denotation: $G(1)$.

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Observation concerning the indexes

Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	1								

Table: First 10 members of $G(1)$

R. P. Stanley observed the following:

Index(2):	0	1								
Value(3):	0	1								

Table: First 10 indexes and members of $G(1)$ written in bases $2, 3$

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Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	1	3	4						

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Observation concerning the indexes

Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	1	3	4	9	10				

Table: First 10 members of $G(1)$

R. P. Stanley observed the following:

Index(2):	0	1								
Value(3):	0	1								

Table: First 10 indexes and members of $G(1)$ written in bases 2, 3

Observation concerning the indexes

Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	1	3	4	9	10	12			

Table: First 10 members of $G(1)$

R. P. Stanley observed the following:

Index(2):	0	1								
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Table: First 10 indexes and members of $G(1)$ written in bases $2, 3$

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Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	1	3	4	9	10	12	13	27	

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Observation concerning the indexes

Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	1	3	4	9	10	12	13	27	28

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Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	1	3	4	9	10	12	13	27	28

Table: First 10 members of $G(1)$

R. P. Stanley observed the following:

Index(2):	0	1	10							
Value(3):	0	1	10							

Table: First 10 indexes and members of $G(1)$ written in bases $2, 3$

Observation concerning the indexes

Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	1	3	4	9	10	12	13	27	28

Table: First 10 members of $G(1)$

R. P. Stanley observed the following:

Index(2):	0	1	10	11						
Value(3):	0	1	10	11						

Table: First 10 indexes and members of $G(1)$ written in bases $2, 3$

Observation concerning the indexes

Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	1	3	4	9	10	12	13	27	28

Table: First 10 members of $G(1)$

R. P. Stanley observed the following:

Index(2):	0	1	10	11	100					
Value(3):	0	1	10	11	100					

Table: First 10 indexes and members of $G(1)$ written in bases $2, 3$

Observation concerning the indexes

Index:	0	1	2	3	4	5	6	7	8	9
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Table: First 10 members of $G(1)$

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Index(2):	0	1	10	11	100	101				
Value(3):	0	1	10	11	100	101				

Table: First 10 indexes and members of $G(1)$ written in bases $2, 3$

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Index:	0	1	2	3	4	5	6	7	8	9
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Table: First 10 members of $G(1)$

R. P. Stanley observed the following:

Index(2):	0	1	10	11	100	101	110			
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Table: First 10 indexes and members of $G(1)$ written in bases $2, 3$

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Table: First 10 members of $G(1)$

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Index(2):	0	1	10	11	100	101	110	111		
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Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	1	3	4	9	10	12	13	27	28

Table: First 10 members of $G(1)$

R. P. Stanley observed the following:

Index(2):	0	1	10	11	100	101	110	111	1000	
Value(3):	0	1	10	11	100	101	110	111	1000	

Table: First 10 indexes and members of $G(1)$ written in bases $2, 3$

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Table: First 10 members of $G(1)$

R. P. Stanley observed the following:

Index(2):	0	1	10	11	100	101	110	111	1000	1001
Value(3):	0	1	10	11	100	101	110	111	1000	1001

Table: First 10 indexes and members of $G(1)$ written in bases $2, 3$

Solution to the Stanley sequence

Theorem:

The members of the greedy sequence $G(1)$ are the integer numbers which ternary representation does not include a digit 2.

Proof of the theorem

Criteria:

- Members of the sequence described by the theorem satisfy the condition of $G(1)$, for whichever three terms.

a_a and a_b don't include a 2, $\Rightarrow$ $a_a + a_b$ has at least a single 1. Therefore it is different than $2 \cdot a_c$.

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Proof of the theorem

Criteria:

- A lexicographically smaller sequence satisfying the condition of $G(1)$ should not exist.

By contradiction (a sequence with a general member b_n) we construct a counter example around α, β, b_m, where m is the smallest number for which $a_m \neq b_m$ and $\alpha, \beta \in G$.

Constructing a formula for the members of $G(1)$

Method of construction:

- Writing the indexes in base 2: $\lfloor \log_2 n \rfloor$ divisions by 2.

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Constructing a formula for the members of $G(1)$

Method of construction:

- Writing the indexes in base 2: $\lfloor \log_2 n \rfloor$ divisions by 2.
- Acquiring the separate digits: $\left[\frac{n}{2^i} - \left\lfloor \frac{n}{2^i} \right\rfloor \right] \cdot 3^{i-1}$
- Finalised formula: $a_n = \sum_{i=1}^{\lfloor \log_2 n \rfloor} \left[\frac{n}{2^i} - \left\lfloor \frac{n}{2^i} \right\rfloor \right] \cdot 3^{i-1} + 3^{\lfloor \log_2 n \rfloor}$

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Order of dependency

Method for construction

- Denote:
  $p_1 \cdot a_1 + p_2 \cdot a_2 + \ldots + p_{t-1} \cdot a_{t-1} + p_t \cdot a_t \neq 0 \Rightarrow O(G) = t$

Example:

Equation of the Stanley sequence:
$a_a + a_b - 2 \cdot a_c \neq 0 \Rightarrow O(G) = 3$

Sequences for which $O(G) = 2$

Equation from $O(G)$: $p_i \cdot a_i + p_j \cdot a_j \neq 0$

p_i, p_j are of the same sign:

$$\Rightarrow G = \{0, 1, 2, ...\infty\}$$

p_i, p_j are of opposite signs: $j > i$

WLG $\Rightarrow a_j \neq -\frac{p_i}{p_j} \cdot a_i$

$$\Rightarrow a_n = n + \left\lfloor \frac{n \cdot \gcd(p_i, p_j)}{p_i} \right\rfloor$$

Example:

$-\frac{6}{-3} \cdot a_i = \frac{6}{3} \cdot a_i \neq a_j \Rightarrow$: first members are: 0, 1, 3, 5, 7, 9, 11.

Sequences for which $O(G) = 3$

Equation of the sequences: $p_i \cdot a_i + p_j \cdot a_j + p_k \cdot a_k \neq 0$

Stanley-like sequences:

- Equation from $O(G)$: $a_i + a_j \neq -p_k \cdot a_k$
- $a_0 = 0$ and $a_1 = 1$
- $p_k < 0$ or $G = \{0, 1, 2, ..., \infty\}$

Sequences for which $O(G) = 3$

Interesting cases of *Stanley-like* sequences:

$p_k = -1 : \Rightarrow$ a sequence "avoiding" the Fibonacci one (0, 1, 2, 4, 7, 10, 13, 16, ...):

$$a_n = n \cdot \left\lfloor \frac{6}{n+4} \right\rfloor + (1 + 3 \cdot (n-2)) \cdot \left\lceil \frac{n-2}{n+4} \right\rceil$$

$p_k \leq -2$ with additional condition a_k: $G = \{0, 1, 2, ..., \infty\}$

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Greedy sequences for which $O(G) = 4$

Stanley-like sequences with additional condition for which $O(G) = 4$

Equation of the sequences: $a_i + a_j + a_k \neq -p_n \cdot a_n$

- $p_n \in (-\infty; -2) \cup (0; \infty) \Rightarrow G = \{0, 1, 2, ..., \infty\}$
- $p_n = -1 \Rightarrow$:

$$a_n = \left(2^n - n - \left\lfloor \frac{1}{n+1} \right\rfloor\right) \cdot \left\lfloor \frac{12}{n+8} \right\rfloor + (1 + 7 \cdot (n-3)) \cdot \left\lceil \frac{n-4}{n+8} \right\rceil$$

- $p_n = -2 \Rightarrow$:

$$a_n = n \cdot \left\lfloor \frac{12}{n+8} \right\rfloor + (1 + 2 \cdot (n-3)) \cdot \left\lceil \frac{n-4}{n+8} \right\rceil$$

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Review of conducted simulations

Hypothesis

Every member of a given *Stanley-like* sequence is a number which can be written in a specific number sequence and read in another one.

Explored ranges:

- $O(G) \in [2; 25]$
- $p_n \in [-1000; -1]$
- $x \in [2; 100]$, where x is the base of the "writing" number sequence.
- $y \in [3; 100]$, where y is the base of the "reading" number sequence.

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Summary of original results

- Stanley sequence:
  - Alternative proof of the Stanley theorem;
  - An explicit formula for the construction of terms of $G(1)$;
- *Stanley-like* sequences:
  - $O(G) = 2$ - complete analysis including a proof and a construction formula;
  - $O(G) = 3$ with additional condition - complete analysis including proofs and construction formulae;
  - $O(G) = 4$ with additional condition - complete analysis including proofs and construction formulae;
- Computer simulations: refuting of the hypothesis in the explored ranges;

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Thank you for the attention!