Numbers as Properties in Logicism

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Abstract

Since Russell's paradox was pointed to Frege, the logicism movement with the exception of the neo-logicists has been largely abandoned. In this paper, Frege's semantic system is revisited and his arguments are refuted to show that numbers are not objects but concepts, and more specifically - properties. It is shown there are needed to distinguish between entities and a logical definition of such distinctions is given. Singular terms and predicates are also revised to show that the new system can benefit from Hume's principle and can enjoy the results from Frege's theorem.

1 Introduction

Gottlob Frege is rightfully regarded as one of the most influential philosophers of the Western tradition both because of his foundation of the analytic movement and his work in the philosophy of mathematics. In 1884, he published his *Die Grundlagen der Arithmetik*[1] (*The Foundations of Arithmetic* in English) in which he argued against Stricker's and others' psychologism and went on to establish his own theory culminating in a platonic view that numbers[2] are objects[3][3].

In addition, Frege was not only a Platonist in terms of his philosophy of mathematics but also a Logicist, hence he attempted to show that arithmetic is reducible to logic. His definitions and metaphysical system led to an important result - Frege's theorem - showing that the second-order axioms of Peano arithmetic is a result of second-order logic and Hume's principle. While Frege dismissed the principle in the *Grundlagen*, his own attempt at logicism came to an end after Russell discovered the paradox that bears his name as it was the direct result of one of Frege's axioms known as Basic Law V.

After the events around Russell's paradox most philosophers abandoned logicism and the movement has mainly been explored in the form of Neo-Fregeanism (Neo-Logicism) by authors such as Hale and Wright[5], MacBride [7], Sullivan and Potter[10], and others. Neo-Logicism is usually understood as the usage of Frege's metaphysical and semantic conclusions about numbers in combination with abstraction principles such as Hume's.

In this paper, we offer a more radical revival of the logicism movement by reconsidering Frege's system and conclusions. The central idea behind the pa-

[1]Shortened to the *Grundlagen* in the remainder of the paper.

[2]As in the *Grundlagen* we will discuss only the cardinal numbers in this paper. The words 'number' and 'numbers' refer only to them throughout this paper.

[3]At this instance, the word 'object' is used in the sense that Frege meant it. Details follow in further sections where objects are compared to concepts explaining that objects are those entities who are referred by singular terms according to Frege. His metaphysical reasoning is discussed in another paper - *On Concept and Object*[4].

per is that Frege's arguments against the view that numbers are properties[4] fail to convince one that numbers are not properties. This is the main subject of the first part of Chapter 3 which follows a detailed introduction on Frege's object and concept distinction and the nature of properties in Chapter 2. In the latter parts of Chapter 3, a positive account on numbers as properties is provided accompanied by a metaphysical explanation of how distinction between entities happen in a logical (in terms of logicism) manner relying on Frege's *falling under* relationship. In Chapter 4, it is discussed how the conclusions of this paper does not contradict Hume's principle and are able to use it to also achieve Peano arithmetic as dictated by Frege's theorem. Finally, in Chapter 5, the results of this study are summarised and discussed.

Here is also a good point to discuss the motivation behind any attempts at logicism considering the destructive power that Russell's paradox exercised on the movement. While Platonism and formalism are usually enough for the purposes of mathematicians, philosophers and especially those concerned with the epistemology of mathematics are often facing great difficulties with reconciling these with various philosophical issues. Whether mathematics is reducible to logic or not, most philosophers and mathematicians will agree that mathematics is reliant on logic even in terms of Natural deduction or other proof systems. It follows that any philosophy of mathematics will have to include a philosophy of logic. One motivation for logicim, hence comes in the form of a loose interpretation of Ockham's razor - striving to make logic the only thing in need of an explanation in order to explain mathematics.

Now, in order to claim that natural numbers are properties, one has to first understand what properties are especially under Frege's system and to this there is the need to understand Frege's system itself - namely the terms object and concept as well as their semantic and metaphysical implications. This is what follows in the first part of Chapter 2.

2 The metaphysics of numbers

Frege's analytic philosophy relied on a connection between the metaphysics being studied and its semantic implications. Apart from numbers Frege develops a binary view on the entities in the universe in *On Concept and Object* concluding that everything is either an object or a concept and there is an easy way to find out what it is by looking at how we refer to it in a well-formed sentence[4].

Here, we will discuss the different entities under the Fregean system and will provide examples for both objects and concepts. While Frege, gives most im-

[4]Here, the word 'properties' is not used to describe only those properties which Mill considered throughout his work. Properties such as validity, cardinality, and others are considered in addition to the usual colour, solidity, etc. More detailed account on that follows in Chapter 3.

portance to the semantic distinction it is equally important to consider the metaphysical insights that led to the semantic implications.

2.1 Brief classification of entities in the *Grundlagen*

In the introduction of the *Grundlagen*, Frege introduced the terms *object* and *concept*. It is evident that the distinction between the two entity types is relevant given Frege's own third principle: "The distinction between concept and object must be kept in mind." [3] Let us now move to explain what are the differences between the two and examine Frege's proposed definition. First, it is important to note that according to Frege an entity cannot become an object after being concept or vice versa while remaining the same: "As concerns the third point, it is a mere illusion to suppose that a concept can be made into an object without altering it." Second, an entity must be either an object or a concept.

In order to explain the former point we have to consider how are entities categorized as either objects or concepts. The distinction is *semantic*. According to Frege, something is an object when it is referable to by a *singular term* and, by extension, the definite article. That being said, it is important to note that the emphasis is on *singular term*[5] and in natural language the definite article is not always needed. Hence, the definite article always implies a *singular term*, but the term can also be achieved without it. An example will be something like a name: "Atanas is a person" (provided there is enough context so that 'Atanas' refers to one person only). An example is also useful when introducing how concepts do not rely on singular terms. Consider the following sentence: "Apples are fruit." Here we are claiming something about a collection of sorts or more precisely about its members - in our case these are all *individual* apples. Because it won't be possible to distinguish a single entity to which I am referring, we conclude that here we speak about something related to the concept of apples. If, on the other hand, I say something such as "the apple in my hand is round" I am inquiring about a single entity thus an object apple. Under Frege's system, these judgments are the result of using a predicate ('are fruit') in the first instance and a *singular term* ('the apple') in the latter one.

This distinction presents the need of some considerations. First, we have to understand whether concepts are indeed unreachable by a singular term. We can, for example, propose that there is a singular term in the first sentence equivalent to "the concept of apples". However, according to Frege because of the definite article this would actually be a name which refers to a non-spatial entity that is not a concept but an object since the former cannot be referred to by a singular term. If we attempt to remove the article and say just "apples" as

[5] Frege's definition of *singular term* can be thought of the parts of a sentence that are referring to objects. It is argued in this paper that both concepts and objects can be referred to by a *singular term* and, hence a new definition is explored - *singular term* is an expression that is left once all the predicates are removed from a sentence.

in our original example this is indeed no longer a proper name of an object but is neither of a concept. Frege calls such semantic entities concept words which, according to §51 from the *Grundlagen*, "just designates a concept". There is a lot to say about this metaphysical decision and it is discussed at length in the same section but for now we will accept Frege's distinction while abstaining from the conclusion that we cannot inquire about specific concepts because of the lack of a proper name.

Next, we have to consider an interesting trait about concepts - their ability to have entities *fall under* them[6]. The basic idea behind this function is that we can observe sentences about concepts providing insight about objects. If this is true there will exist a relationship between the two entity types. Consider our initial example - "Apples are fruit." As we've discussed there is nothing about this sentence *directly* connected to individually distinguishable apples, hence objects (at least according to Frege's system). However, we can also agree that if I am to construct a sentence about an existing object apple, the apple will definitely be a fruit. We attribute this to an object apple *falling under* the concept of apples. This is because the clear implication about an object referred to by a *singular term* definitely possessing a property revealed through the predicate in the sentence with a concept word indicates there is a relationship between objects and concepts.

This definition (or more precisely explanation) prompts an important question about whether it is possible for an entity to fall under an object. For Frege the answer is absolutely no. He thinks that this cannot be the case because we refer to objects by using the definite article (which constitutes a proper name) which will result in any sentences about the object providing insight about the individual entity and nothing more. This is, perhaps, the most intriguing part of Frege's classification system. It provides the opportunity to reconsider the somewhat controversial distinction about the nature of a proper name and a concept word based on the definite article being present exclusively in the former as discussed earlier and found in §51[3].

It is true that Frege's own explanation of a concept word - "that it just designates a concept" can prove controversial if not properly distinguished from a proper name in terms of more than semantics or at least additional argumentation about why the semantics is of such importance here. First, I agree with Frege when he points that a sentence where the subject is attained by a definite article provides insight about a single entity exclusively. If we have that "*the* apple is round" we have knowledge about a single object and nothing more[7].

[6] One can notice the use of the word 'entities' instead of simply 'objects' in the former sentence which is discussed in a bit.

[7] It is important to point out that under our system the correct way to say something conveying information about all apples being fruit will be using a concept word and a quantifier like "All apples are fruit." even if one argues that "The apple is a fruit." conveys the same meaning.

However, it should not be assumed that something that can convey information about entities other than itself (such as a concept) cannot under any circumstances convey information about itself exclusively. An interesting sentence to consider would be "The concept of A is not an object" which one may argue is different from " 'the concept of A' is not an object". An answer to this may be that " 'the concept of A' " is a name of "the concept of A" but because names are objects under Frege (because of the definite article) a name of a name would be hard to justify. This is so because Frege does not allow for objects to fall under other objects which would be a direct implication if we want to keep *falling under* the only relationship. Hence, because we have established the use of only the *falling under* relationship and there is a demonstrated connection between an entity and its name given that the name refers to the entity, the entity in question *falls under* the name. And because the entity is int this case a name itself this constitutes a problem for the original system which can only be solved by allowing for another relationship such as an extension. In any case it is reasonable to believe that if we are wrong Frege has to present additional argumentation.

If we are to continue the above reasoning and argue that the semantic singular term distinction is not enough to differentiate between an object and a concept we arrive at a new problem. Namely, how can we distinguish an object from a concept and how can we say if something is an object or a concept? The answer to this is vital as it will be useless to have a system without means to use it given our goal is to inquire about the nature of numbers. We have shown some groundwork to challenge the view that something being definite makes it an object. The reasonable alternative is to consider the opposite principle - something being indefinite makes it a concept. Of course, this is not the perfect system because under the Fregean way we always knew if something is an object or a concept by checking the article. We can explain Frege's system as such that objects are always referred to by singular terms and concepts by predicates. In our system, we can only say with certainty that something is a concept if it is referred to by a predicate (or is at least indefinite). Now, we can categorize entities only if we have an example where the subject is indefinite but not if it is concrete.

To build upon the proposed semantic distinction let us consider its implications. As we discussed earlier if the definite article is used in the sentence we inquire exclusively about the subject and if not that is no longer the case as noted by the trait *have things fall under* present in concepts. We can thus propose that the difference lies in this trait specifically and, hence with sentences conveying things about something other than the subject. While this does not solve the problem from the beginning of the paragraph, we only need to consider whether numbers are objects or concepts for the purposes of this paper.

2.1.1 Examples of objects

Now, let us consider some examples of entities that are either objects or concepts and provide some justification for their categorization under our Fregean system. Let us also use this section to revisit the point that something has to either be an object or a concept even with the small revision we brought to Frege's original system. This is the case because the new condition of separation which comes down to which things are concepts relates to the ability of the semantic entities that refer to them to provide information not only about a singular entity.

Most if not all *spatial* things are objects. Here I use *spatial* to refer about things in possession of certain properties like colour, solidity, smell, and others. Such objects are individual stones, dishes, apples, etc. In order to make this more rigorous it may be useful that the exact collection of properties needed is such that the entity is perceivable by the senses (we will return to these properties in a bit). This means that the Sun is spatial but an argument is not. An image of the argument such as it being written on paper is another entity in itself just as a picture of the Sun on paper is not the Sun itself (though in that case they are both spatial). Of course, this is not enough to say that something spatial is an object. An interesting example will be a crown of a tree. Let us consider the crown being green. We will have to admit that the individual leaves will be green as well yet the crown is perceivable. Or maybe it is not. We can provide an empirical test such as the following: if the perception of something by each sense is acquired by the perception of its building blocks by the same sense then there is not a distinguishing perception to promote the hypothesis that the entity in question is spatial. Then the consideration of the previous point depends on whether we consider both the smell, sound, look, touch, and taste of the crown equivalent to that of the individual leaves.

Here, we leave this pondering as they are not too relevant to the nature of numbers given we are just giving examples of objects and concepts. One point we will make here is that, just like Crane and Mellor point in their paper, such definition of spatiality when connected to physicalism and materialism may lay grounds for the inclusion of psychologism [2]. In order to avoid this, we can claim that while individual perceptions can be and are subjective, all of them are based on set of senses such that the element of the set are not subjective[8]. This is already too broad, however, so it is a nice point to step back and return to giving examples.

Let us, now, note that non-spatial entities can still be objects. Individual sentences are not spatial but are objects because in a sentence about them such as

[8] That being said, they can still lead to subjective conclusions. We claim that *perceptions* will rely on certain senses such as sight or smell and not that the employment of sight and smell towards a single entity will result in the same perception for more than one observer. The important thing is that both of them will experience a *perception*.

"this sentence is non-spatial" does not offer any information about anything but the subject[9]. The same point can be made about other semantic entities such as words. If one claims that these don't have meaning without symbolic representation (which will require perception) we can still use arguments as examples - "This argument is non-spatial".

2.1.2 Examples of concepts

Because we will be claiming that individual numbers are distinct concepts, it is concepts that interest us more. First, we note that it is hard to provide an example of a spatial concept without having clarified the specific nature of spatiality and, hence being subjected to psychologism in terms of subjective perception. We can, on the other hand, provide non-spatial or at least more abstract examples. The most intriguing for our purposes are property types. Here I am using the terminology property type and token as discussed in a paper by Alex Oliver in which he discusses the nature of properties [9]. An example of a property type will be colour and the tokens will be red, blue, green, etc. Property types can also be a bit more abstract as for example validity (of an argument) such that the tokens will be 'valid' and 'not valid'. Properties are evidently concepts which is made clear by a sentence such as "The apple is green" which tells us something not about the 'green' token but about the individual apple in question. One can see that properties are closely related to the *falling under* trait which can, in fact, be turned into a property token itself if rewritten as 'can have entities fall under' with the other token of the type being 'cannot have entities fall under'. This makes it evident that there exists a relation between both the notions of concepts in general and properties and sets as in the mathematical field of set theory. Just like them, which under Frege are also concepts for exactly the same reason, concepts (including properties) are independent of the mind (in terms of psychologism) and existing entities as discussed in the *Grundlagen*. It is obvious how sets are concepts as many times it is something about their members that defines them in the first place. However, sets in mathematics have cardinality as a property which does not convey any information regarding something but the set which gives further grounds as to speculate that Frege's initial singular term distinction might have been flawed. Here, it is important to note what we are suggesting is the relationship between a set and its cardinality. The latter means nothing without the former and the former will be altered as a result of altering the latter.

Let us now return to properties as they are central to our discussion and we will argue that numbers are properties. It is good to clarify the distinction between concepts and properties - a member of the latter is always a concept but only those concepts that are used to distinguish between things are properties. In order to inquire about the precise condition of properties it is important to

[9] At this instance, we refer to the word 'sentence' in terms of sentence types exclusively. Hence, we do not consider the actual writing down of a sentence on this page but merely a specific sentence type "this sentence is non-spatial" falling under the concept of a sentence.

consider some classes (so as to not use the word ‘types’) of properties.

2.2 Classes of property types

It will be useful to return to our original two examples when we introduced properties - colour and validity. It is intuitive to presume that there is some kind of difference between the two types as it won’t be very intuitive to talk about the validity of a leaf or the colour of an argument. However, this may or may not be the case.

We have two options ahead of us. We can consider that properties have **restricted domains** or that they have **unrestricted domains** whereas we will define the domain of a property type as the collection [10] of entities that fall under the concept of the property in question. This means that under the first option not all entities necessarily have a colour or validity and under the second they all do. We will consider the options one by one.

2.2.1 Properties having restricted domains

If we accept the intuitive premise from the beginning of the section we are restricting the domain of certain property types. Our example with the valid/invalid leaf and the colorful argument illustrate how this is done for two types: validity and colour. It seems that these restrictions are somehow connected on spatiality so we are already risking entering the realm of psychologism. We can name properties like colour *physical* (though they are still non-spatial) and properties like validity *non-physical*. The question now is what makes a property physical/non-physical.

To solve this we can attempt to revisit our attempt to capture spatiality and return to perceptions guided by Berkeley’s “To be is to be perceived.” principle. If this is acceptable we can simply say that physical properties are those that are *sufficient* for a perception of some kind to be obtained. It is necessary to go with the easy epistemology when speaking of the logicist program so we will just point that if we have a given colour we have some sort of a perception while if we know whether it is valid or not does not result in the same perception [11]. This is what we will use to restrict the domain of a certain property on the physical/non-physical axis.

However, this is not a perfect solution. This is because there are entities of which we can say that are for example colorless such as a drop of water or the atmosphere [12]. But if that is the case why not say that the argument we were discussing as an example is colorless as well?

[10] Not necessarily a set although the difference may be negligible.

[11] Such that the last word is used in terms of the senses and not necessarily of any other understanding of the examined entity.

[12] Here whether these are objects or concepts is not relevant as concepts can fall under concepts too.

2.2.2 Properties having unrestricted domains

Following this trail we may arrive at the conclusion that properties are not restricted in domain. We can still make use of the physical/non-physical distinction without the need of restricting domains by saying that all things that are non-spatial for example are colorless. Here, validity is the more troubling type. If we consider everything that is spatial invalid we arrive at a bit of problem in the very foundations of classical logic. On one hand, invalidity certainly exists and according to the law of excluded middle if something is not valid it must be invalid. On the other, it is important to point that an argument is valid only if it meets certain conditions such as consisting of well-formed sentences. It is essential, therefore, that we admit a reason such as this to be the same for branding a leaf or an apple as invalid[13]. If that is not the case and we does not have a reason for invalidity (equivalent to saying that something must be an argument to be invalid or must be an argument to breach a condition of validity) we are either restricting the domain or breaching the aforementioned logical law.

Thankfully, for the purposes of the logicist program such decision as to the nature of the domain of property types is not needed. However, the physical/non-physical distinction turns out to be vital when considering Frege's arguments from the *Grundlagen* against our view that numbers are properties. This is so because several of the arguments pointed specifically at Mill's view of properties (only *physical* ones are considered by him) rely on *spatiality* and are easily refuted once we admit things like validity to be properties as well. To sum up, the term *physical* refers only to a class of properties which if possesed by an entity allows for an agent with knowledge of that property to form a partial sensory perception. Hence, 'being *physical*' as a property of properties is dependent on *spatiality*. In order to preserve the use of the term we will accept the condition of being sufficient to form a partial sensory perception for something to be called a physical property and will add that for something to be considered spatial a partial sensory perception is a required condition. Here, it is important to note that it is needed to be a sensory perception which can be subjective. This gives us grounds to say that because we do not care about the exact perception as long as it is a sensory one (in case others even exist) we do not fall for some kind of psychologism as cautioned by Crane and Mellor[2][14].

2.2.3 Binary property types

Finally, before moving to the discussion of numbers as properties, we consider a class of properties we will call binary. These are such property types that have exactly two tokens such as validity as we have explained earlier. Usually, they

[13]In order to clarify it is useful to mention that we are not allowed to dismiss this as a categorical error because this will be the definition of restricted domain. Later on, it is established that for the purposes of logicism it is not necessary to accept that number properties have unrestricted domains.

[14]That is, because we only care if *it is possible* for such a perception to be acquired which is still objective and binary.

are formed because of the law of the excluded middle as we have shown. The reason we are discussing them is to point out an important principle. If we have an individual property token p it can not be the sole one in a type because we can always make a new token equivalent to $\neg p$. We can then introduce a property of property types called cardinality which is the same as cardinality of sets but instead of counting members counts tokens. This is of vital importance because it gives us the difference between a set and a property type - the cardinality of the latter is larger or equal to 2. This is pointed by Frege in §29 and §30 from the *Grundlagen* in which he suggests that if a number is ascribed to everything it cannot be a property because properties are used to distinguish things apart[3]. An example might be useful to illustrate the point. Consider a property type such that all entities that have the property fall under one specific token (a) of the type. Then, by definition, the token that is $\neg a$ must exist if only to provide meaning to the expression "x falls under a" because if all things fall under something the property does not serve its purpose - to distinguish between entities[15].

We have now arrived at a metaphysical view regarding properties - they are concepts used to distinguish between things such that a concrete number of entities fall under them and this number is larger than 1 and smaller than the number of all entities in existence. In the latter sections of this paper, it will be shown how properties are used to distinguish between things as well as how they are referred to in a sentence. This will be essential when arguing that predicates refer only to concepts but singular terms can refer to both objects and concepts. Now that we have explained the metaphysics behind Frege's classification of entities we proceed by examining the possibility of numbers being properties and his arguments against that view.

3 Numbers as properties

In this section, we consider the view of numbers being properties and how it relates to the logicist's cause. We explore Frege's arguments from the *Grundlagen* against the view and discuss their flaws. We continue by explaining what kind of properties numbers can be and what would such a view mean under the Fregean object-concept system.

First, let's turn the attention to the arguments against the view.

[15]It may be of interest for the reader what happens for properties that supposedly apply to every entity. The answer from this system will be that such an entity which has everything fall under it is a concept but not a property. One specifically important such property is "being identical with itself". Our system says that the real property is "being identical with x" where x is the entity and x changes depending on the entity. If the system is accused of violating Ockham's razor by introducing an infinite number of such properties, it can be argued that they exist just as perfectly and the general "being identical with itself" is the true intruder.

3.1 Frege's arguments from the *Grundlagen*

In his book, Frege details the distinction between the two types of entities in his system - objects and concepts. Moreover, we understand that everything that is not an object must be a concept. Frege discusses several entities such as properties and sets (of objects). He gives arguments against seeing numbers as either of these in order to convince us that numbers are objects. In the case of properties, the laying out of the arguments begins at §21 from the *Grundlagen* when Frege introduces the problem related to Mill's view that numbers are *physical* properties. When first he considers the problem, the example he chooses is colour: "It is natural to ask whether we must think of the individual numbers too as such properties, and whether, accordingly, the concept of Number can be classed along with that, say, of colour" [3].

Here it is worth spending a moment to reflect on our goal when attempting to see numbers as properties. It is important to note that the word *physical* may have a huge effect on both the usefulness of Frege's arguments and our own attempts. While there exists previous work on whether Frege was too quick in discarding Mill's view that numbers are physical properties such as papers by Andrew D. Irvine [6] and Brendan P. Minogue [8], our dissatisfaction with Frege's arguments is more general.

Because one of the practical differences between objects and concepts is that no entities may fall under objects, it seems we must also consider *non-physical* (by extension this means not exclusive to spatial objects) properties. One should be willing to admit that some arguments are valid and some are not allowing us to make a property of *validity* as discussed in previous sections. If we restrict *validity* to arguments by definition and agree that arguments are *non-spatial*, we can conclude that *non-physical* properties exist[16]. Hence, because, by definition, arguments fall under *validity* these properties are not objects but concepts according to the Fregean definitions. We shall, therefore, keep in mind that arguments related to *physical* properties exclusively may not be of great help in proving that numbers can not be properties in general.

In his article, Irvine summarizes 8 arguments in total presented first in the *Grundlagen* and argues that all of them are to different degrees unconvincing in disproving Mill's view (of numbers being *physical* properties)[6]. For our purposes, a different argumentation is needed as reflected by the previous point. It is useful to characterize the arguments Irvine considered into three groups - related to numbers as properties in general, related to specific numbers as properties, and discussing *physical* properties exclusively which, as we explained earlier, does not necessarily concern us.

[16] *Physical* and *non-physical* properties are all *non-spatial*. The difference lies in whether they are sufficient to form a sensual partial perception of an entity falling under them. That being said, this implies that the entities that fall under *physical* properties are *spatial*.

In the last group falls **Frege's last argument** - that numbers unlike colour, solidity, and other *physical* properties may also be applied to *non-spatial* entities (an easy contemporary example might be cardinality of a set) (§24)[3]. As discussed there exist *non-physical* properties like our example of *validity*. This gives us grounds to disregard this argument but in attempt to not be rather too quick it is important to note the possibility that something have to refer to at least one *spatial* entity in order to be a property (no matter how unintuitive that sounds). A quick construct may be based on a predicate like 'is not red' extrapolated to the property token 'red' from the type 'colour'. In other words, we can quickly make the desired properties by creating a binary property type based on the existing token from the initial type.

Next, we discuss Frege's arguments concerning all natural numbers.

3.1.1 Arguments about numbers as properties independent of the choice of number

The first argument we consider relates to what specifically a property applies to. Frege argues in §22 and later in §25 like Baumann that **physical properties like colour refer to an entity as a whole, whereas numbers require more detailed instruction**. Frege's examples include ascribing both 52 and 1 to a deck of cards dependent on whether we consider the name of the deck as a collection of cards or as a singular pack. Another example is ascribing 1, 24, and a large number to the *Iliad* dependent on whether we see it as one book, collection of songs, or of verses. In his paper, Irvine offers an argument in support of Mill as to whether such a view is flawed [6]. For our purposes, I offer argumentation that all properties including physical ones need further instruction if asked in a form close to Frege's examples.

For example, let's consider the colour of an apple. At first, one may consider this intuitively easy and ascribe the property token of its peel, but it is often that the colour of the inside part is different than that of the peel. One may argue that the peel of an apple is an independent, singular object while a collection of verses is not but because of Frege's own hierarchy of concepts this is not a difficulty. The difference will be that numbers and colours are different order concepts. The only grounds for opposition will be to claim that such collections are always vague enough to be ascribed more than one property token number but that would seem to be a definition issue. One may argue as well that the number one can be ascribed to every entity because it exists but we will discuss this in the next group of arguments.

Additionally, Frege calls into question **the way in which we choose to divide a supposed whole like a deck of cards believing it may be arbitrary** in §22. However, this argument presents a difficulty to itself. If we are free to choose any number and find a way in which to divide a whole entity this needs further justification. If we are somewhat (but not completely!) restricted one may ask what numbers are we restricted to suggesting that the structure of the entity is not arbitrary[17].

Another interesting argument deals with the presumption that ***physical* properties are inherited when a whole entity is divided but a number property won't be**. In §22 an example is given with the colour green. It is suggested that if we ask about the crown of a tree it is green and if we choose to examine its leaf independently it will be green as well. On the contrary, if we assume the crown to be a set, we will have a large number ascribed to it while we are also bound to choose 1 if examining individual leaves. Irvine points out that in this argument the presumption is false citing properties such as voltage (of a system) and freezing temperature (of water) as counterexamples. He argues that these change once we consider only a part of the system in question (such as a partial electric circuit or a single water molecule)[6].

I feel, however, that this is not enough as the chosen property types have tokens that are at least partially numbers. Irvine's example about inquiring about the freezing temperature of water will result in an answer in degrees but still a *number of* degrees. The same is true for voltage of a system. In order to counter this, we may consider an example of our own related to a *non-physical* property type. We will continue Frege's example and use a set. Now let us consider the set of all singletons that exist. The property 'not being a singleton' (different than the predicate 'is not a singleton') is one property token that works. If we are not that strict we can also make examples out of well-formedness as the parts of a WFF are not well-formed themselves. If we want a physical example, perhaps we can use solidity and argue that having a property token from the solidity type requires the containment of multiple molecules. This is so because in order for something to be either solid, liquid or gas it must be composed of more than one molecules. Hence, Irvine's water to single water molecules example (having different freezing temperatures) will now work for say losing the property of liquidness. Later, in §29, Frege offers another example including a sentence which justifies our WFF example but is otherwise same in nature and so will not be discussed.

Let's now move on to Frege's arguments concerning specific numbers.

[17] Here, even if Frege claims that there are many numbers that can be ascribed to a deck of cards but concede that there is at least one that cannot be, he must provide a reason about this which he will only be able to do if there is a numerical information about the deck. This implies that the choice of which numbers we might ascribe is not arbitrary at all and is already using something numerical about the deck.

3.1.2 Arguments about numbers as properties related to specific numbers

Frege offers several arguments of such nature related to the numbers zero, one, and also “very large numbers”.

First, let’s consider the case with 0. According to Frege, **zero cannot be a *physical* property because it can only be ascribed to *non-physical* concepts** [3]. In §46, Frege offers the following justification: “If I say “Venus has 0 moons”, there simply does not exist any moon or agglomeration of moons for anything to be asserted of; but what happens is that a property is assigned to the concept “moon of Venus”, namely that of including nothing under it.” It is clear that this is a problem only if we want numbers to be *physical* properties, but we can even better take the time and establish what the token zero will mean under our system in general. The most obvious choice should be to signify the absence of something. Moreover, this seems to have a precedent with physical properties (not all, though) like colour where we have the interesting token of being transparent that in a way can be interpreted as lacking another colour (even if it conveys additional information for other physical properties of the entity in question).

When we consider the number one being a property Frege has two arguments in attempt to convince us otherwise. First, in §29 and §30 Frege suggests that **the number one can be ascribed to everything**. If this is true and one can indeed be ascribed to any entity it means it is meaningless because as Frege points the point of properties is to distinguish entities from one another. While this is not unquestionable I agree with it being a good enough definition given the logicist program and will try to find another justification. A somewhat intuitive answer will be to use Frege’s own argumentation against him. For example, let’s return to the deck of cards. As Frege pointed out it is not the case that both 52 and 1 are properties of the deck because 1 is property of the pack and 52 is property of the collection of cards. As we do not mind numbers being *non-physical* properties it will be helpful to decide that they convey information about concepts exclusively (more on that later) and in other words are not first-order concepts. Now we can see how numbers are applied to other sets and, hence, the number one is ascribed to singletons just as with cardinality.

This is, in fact, closely related to the second argument Frege has against the view of number one being a property. In §31 Frege argues that when we say something about numbers we refer exclusively to concepts and not to any kind of *spatial* entities which under his system must be objects. This, however, leads to a distinction between being able to individuate and have knowledge of the number one. According to Irvine this presents a problem to the supporters of scientific realism which he then discusses in detail and offers a potential solution. I will not discuss this as it is of little interest to the issue at hand. The same is true for Frege's argument about large numbers which does not criticize the idea of numbers being properties but offers argumentation against this view in case someone thinks arithmetic is based on empirical observation - of which Frege accuses Mill in §7.

We have now arrived at the last of Frege's arguments that awaits discussion. It is again based on one yet not on the concept but the word 'one' itself. In addition, it is not an argument we discussed because it presents too great of a challenge but rather because introduces the dependency between properties and predicates of which we are interested. Frege writes in §29 that while one can say "Solon was wise" in isolation and still have meaning extracted from this, they cannot do the same with "Solon was one". This seems rather strange even on the border of being an assumption. If one agrees that entities may be different than one which we discussed earlier, they can also agree that there exist a subject that is not one. Even in non-formal language a sentence like "the songs of the *Iliad* are not one" makes sense even if it is achieved by the plural. While it may not be his strongest argument, however, Frege shows clearly that he understands that a very strong connection between properties and predicates exist. In the next sections, we will consider their precise relationship and discuss on what kind of properties we want to argue numbers are for the goals of logicism.

One thing we can note here is that we will want to establish that number properties can be ascribed to at least one non-spatial entity without restrictions on whether it is an object or a concept. The reason for this might be extracted from several of Frege's arguments and more precisely how they are specifically phrased against Mill's view of numbers being *physical* properties. In §12-§14 Frege argues against Kant that numbers are, in fact, not *synthetic* but *analytic a priori* truths (*analytic a posteriori* is ruled out in §3). I endorse this view which in order to preserve will bound us to meet the condition from the beginning of this paragraph because of the possibility of someone arguing that if a number property is reliant on *spatial* objects to exist, there exists the possibility that any truth about it is *synthetic*. It is vital that this does not happen as it will undermine the process of reducing arithmetic to logic and nothing else which is in essence the core of our program.

We have now attained a general idea about the nature of properties as Fregean concepts which fall under several types as guided by earlier distinctions made in this paper (i.e. physical/non-physical, binary/non-binary). We have also explored all of Frege's arguments and found them all at least partially unconvincing. Given that considering them objects was ultimately unsuccessful, it may be good to ask ourselves the question - "Could numbers be concepts" and more specifically properties.

3.2 Properties under the Fregean system

First, we have to inquire about how exactly properties work under Frege's system of objects and concepts. We know that properties are concepts because they have entities fall under them. This is the metaphysical nature of properties in general according to Frege but it leaves grounds for interpretation.

Let's note again the justification about the existence of properties — these are the entities by which we distinguish entities. This epistemological claim in itself raises a lot of important questions. Are we using properties to distinguish only between objects or can they be used to distinguish between concepts as well? How do we distinguish between properties? And, perhaps most important, how do we explain the metaphysical process for this epistemic distinction? It is vital to recognize that these questions are fundamental to properties in general, hence not number properties exclusively. We are then obliged to attempt to answer at least the last of them so as to know that our definition of properties is possible under Frege's metaphysical and semantic systems.

Consider the example of a green and a red leaf. How are we to distinguish between them? By using their colour properties. However, it is important to observe the metaphysical relations which allow us to use properties to distinguish between the two leaves. We begin by agreeing that both green and red are colours and so fall under the Colour property. In addition, neither of the leaves is a colour so they do not fall under the Colour property but instead under its tokens — green and red[18]. The question then remains as to how we are able to distinguish between the two leaves.

3.2.1 The inheritance structure

There are two possible explanations worthy of investigation. First, is a way of distinguishing between entities we will from now on address as **the inheritance structure**. We agreed that both green and red are colours because it is clearly impossible to distinguish between the leaves on the basis of one of them being green and the other being red if we were to claim that there is no connection between green and red. From here it is evident that a connection between individual tokens of a property type exists even if the to-be-distinguished entities do not fall under the type itself but under these same individual tokens. This will result in a system under which distinction between entities such that they fall under different types is achieved by these types being themselves tokens of a single property. In our example, the distinction is possible because although one of the leaves falls under green and the other under red, both red and green fall under the Colour property. An illustration may prove helpful in explaining this:

[18]Here, we are observing how entities are able to fall under what we treat as property tokens and not types. This means that when we are claiming that a certain entity has a certain property type like colour or solidity, it in fact falls under one of the tokens of this type creating a two-levels structure. This is discussed in the next section.

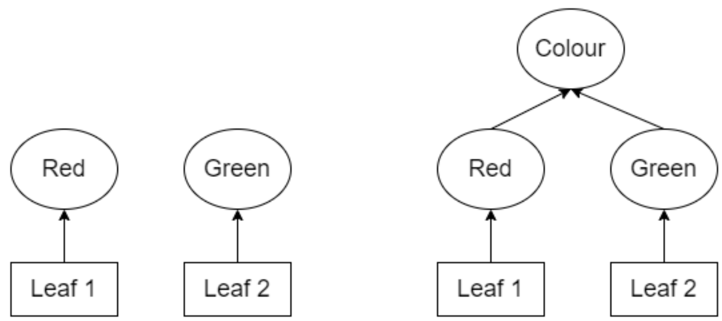

Figure 1: Two diagrams explaining the improper and proper way of using properties to distinguish between entities according to the first approach.

The left diagram on Figure 1 showcases that even if two leaves fall under different properties there is no way to distinguish between them unless the properties themselves fall under a single higher order concept property which is the case in the diagram on the right. Rectangles designate entities (which in these cases are individual leaves and, therefore, objects, circles represent concepts, and an arrow from x to y shows that x falls under y).

This first solution is somewhat intuitive because it is normal to consider that there is some purpose behind the ability of Frege's elegant system for entities to be able to arrange themselves in levels. However, there are vital metaphysical challenges that have to be considered. Namely, how do we distinguish between entities that seemingly fall under the same types. It turns out that this is an especially relevant issue when trying to distinguish between properties. Let's consider our last example. How can we distinguish between green and red? They are both existent, non-spatial, concepts, properties, colours, and their objectivity (in terms of objective/subjective) can safely be said to be the same for both of them[19]. A desperate attempt will consist in making a mathematical distinction presupposing number properties trying to argue about a difference in wavelength/frequency which will simply be followed by an inquiry about how those values are distinguished. More importantly distinction based on the inheritance structure posseses an even greater trait to our program shall we use it to distinguish between numbers - how do we know if the structure itself is reducible to logic and is not simply as described in the beginning of the paragraph - reliant on being 'intuitive'.

[19]Please note that this paper makes a distinction between properties and qualities. We define properties as concepts used to distinguish between entities. Hence, if properties are inherent to an entity's nature is not an issue and so properties can be objective even if qualities aren't.

3.2.2 The 'not' method

To deal with this issue we should investigate the root of the problem. If we were presented with two distinct entities (a and b) such that they *seemingly* fall under the same property types exclusively how can we distinguish between them without intuitively relying on inheritance? We should also strive to make our approach logicistic (in terms of being reducible to logic). One rational solution is to claim that b is not a.

Now, we can see that it is rational to claim that something is different from something else when trying to distinguish the two and in order to distinguish then even if it is not practical. It is important to keep in mind that if we do not adhere to the inheritance structure we cannot claim that the distinction lies in one of the entities falling under a certain type and the other does not which constitutes the distinction because then we face the epistemic challenge of how that information was obtained. Hence, saying that a is not b and is, hence distinguished from b is not the same as saying there is some property type P such that a follows under P and b does not or vice versa[20].

Instead, we can give the following alternative. Let us look at the sentence "a is not b." and observe that the predicate in it is 'is not b' while the subject is 'a'. Hence, we can postulate that 'is not b' is a property token of some property type B with tokens 'is not b' (not b) and 'is b' (b) such that a falls under the former of the two tokens and b under the latter[21]. Now we see that inheritance is a *necessary* truth under Frege's system unless we want to justify the existence of another non-logical relation apart from *falling under*. If that is not the case we fall into the problem of separating things by green and red without adhering to the relationship that green and red are colours (because we can't easily reduce this to logic[22]) and without saying that red is simply not green. Therefore, in order to fix the intuition gap in the inheritance structure we have to add the condition that only binary property types are allowed to be used in the distinction process.

Finally, we are able to offer a logical definition of distinction: An entity x is distinct from an entity y if and only if x falls under a, y falls under $\neg a$, and both a and $\neg a$ fall under A[23]. This definition is illustrated on Figure 2:

[20] This raises an important point about why we don't simply rely on colour exclusion as if something is red all over it certainly is not blue. It is vital to understand that what we are claiming here is that even if being red all over is the reason about why an entity falls under "not blue" it will always fall under "not blue" when distinguishing it from something falling under "blue" because of the binary cardinality rule.

[21] Note that here we are not permitting haecceities on the grounds that they violate the binary cardinality rule given that there is not an entity not identical to itself.

[22] This is meant not in terms of logic providing existential truths by itself but conditional statements. This is clear in the logical definition of distinction that follows.

[23] The last part of both properties falling under A is needed because logic is not existential. If we were able to distinguish between entities without this being the case it would have been solely based on logic which would constitute a problem.

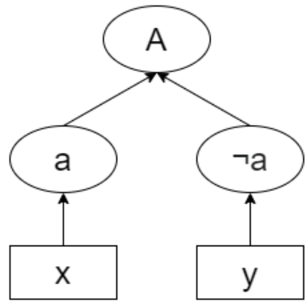

Figure 2: A diagram illustrating the method of logical distinction.

Figure 2 uses the same notation in terms of shapes designating different metaphysical entities as Figure 1. Now, let us move to a discussion on some arguments in favour of the view that numbers are properties.

3.3 Positive account of numbers as properties

First, let us explore what will be the intuitive structure of such properties. We begin with the concept of Number under which all numbers fall. So far we are not breaching into uncharted territory and are in agreement with Frege. However, for him numbers are objects and so the Number concept is a first-level one. As shown in the previous section, the arguments behind this choice are not enough so let's consider it is at least a second-level concept which will make the entities falling under it (numbers) concepts.

We can now form the first property which we will be considering. The property type is Number and the tokens are all individual natural numbers. Here, we are exposed to a possible motivation as to why we should consider numbers as properties rather than objects. This comes down to an eventual definition of numbers as properties, and more specifically the way of distinguishing between different numbers. Of course, Frege offers a definition of numbers suggesting that the number of F's is the *extension* of G such that G is *equinumerous* to F[3].

It is clear that this definition resembles the language we would have used were we to believe that numbers are properties. In order to consider them objects, Frege was forced to adopt the ideas[24] of 'extension' and 'being equinumerous'.

The second of this is closely connected to Hume's principle of one-to-one correspondence, so we will leave it for now and focus on 'extension'. If we do not restrict ourselves to numbers being objects it is much easier to rephrase Frege's

[24] Not necessarily in the psychological sense in which Frege uses it. I am not claiming that they are subjective entities (in accordance to Frege's first principle, but that they are unnecessary which is to be understood as not abiding by Ockham's razor.

definition to "the number of F's is the number property of G such that G is equinumerous to F"[25]. However, this is clearly not enough for several reasons. First, we are referring to a number property when trying to define number. Second, it is not clear if properties can be used to create a one-to-one correspondence given that examples are usually about objects. It is thus clear that any potential definition of numbers that regards them as properties will require a deeper discussion on the Number property type and the individual number property types.

3.3.1 Numbers as needed to distinguish between entities

Let's return to Frege's arguments against that view for a moment. When he was claiming that physical properties are inherited, he noted that, like properties, numbers are usually used in adjectives. Just like we say that "a desk has green drawers" we say that "it has five drawers". After presenting his argument, Frege concludes that it is better to rephrase the latter sentence as "the number of its desks is five" so as to avoid mistaking them for properties. But this raises an important question. Are numbers ever necessary to distinguish between two entities?

Because we use properties to distinguish between entities we could have a foundation to regard numbers as properties if we are able to point a case whereas a number or numbers are needed in order to distinguish between the entities in question. Let's imagine the following example - Two bookcases formed by placing identical shelve boxes on top of each other. Now, let one of the bookcases be formed of five shelves and the other of four. How can we distinguish between them?

For the sake of argument, we restrict ourselves to answers different then their number property so as to give Frege a chance. There are two main answers that need to be considered. First, we have to look at people pointing at a number property "in disguise". For example, suppose someone is trying to point out our bookcases are of different height or mass. We can ask them how that is the case. In order to answer they will either have to provide numbers in general like "one of them is 2.5 meters tall and the other is 2 meters in height" or will have to point out a comparison relationship like "the one to the left[26] is taller".

As evident both of this approaches *rely* on numbers in some sense as comparison is valid over numbers exclusively. Here, our job is to show the property hierarchical structure so as to make sure that we are indeed distinguishing by a property which will grant our system the right to remain important. Let's

[25]This implies that we are abandoning the strict semantic rule that only objects are referable to by singular terms as discussed in Chapter 2. A detailed account on the definitions of predicates and singular terms is offered in Chapter 4 where we discuss Hume's principle.

[26]Position works just like mass and height as it requires the setting up of agreed directions that are, hence subjected to numbers, in order to even mean anything.

assume that there is such a property type as height (which is indeed the case). Now, we should inquire about its tokens because we can distinguish entities by properties without caring deeply about the tokens only when the types are binary which height is not as we are claiming that both bookcases have some height.

As we attempt to answer what is the height of something we are presented with several options. First, there is to answer relative to another entity which we will cover in a bit. Then, one is left with the other option - providing a value (whether it is an approximate or an exact, a wrong or a right one) and a unit. In accordance with our previous example, such is the role of '2.5 meters' and '2 meters'. It becomes obvious that the token of the height property that is possesed by the bookcase is somewhat related to numbers. In addition, this tokens are properties as well because the bookcase does not simply fall under height as in being a height token or as having a height but as being exactly 2.5 meters in height.

Now, there is the question of the unit. What exactly is a meter, for example? The accepted answer will be that it is *defined* as the *equivalent* of a one-millionth of the Earth's quadrant[27]. However, we should ask ourselves what does being *equivalent* mean and will soon arrive at the conclusion that we are concerned with the length of the quadrant. Here, we cannot longer use numbers because saying that a measurement property token is a number without an unit is meaningless as it is not objective. A more plausible version will be to conclude that the definition required that the length of the meter is similar to the length of the other entity. However, every token of the length property type will contain a number in itself, and so the unit is too dependent on numbers. This shows that the attempt to distinguish between the two bookcases by providing a property type about measurement is not enough because both bookcases will posses a token of the type and the token will be completely dependent on numbers as all units are dependent on numbers even if their number is unknown.

We can also attempt to withhold a number from our answer to the question about how is there a difference between a measurement property shared by both bookcases by answering that, for example, one is taller than the other. However, this relationship instead of a property answer achieves virtually nothing as we can again inquire about this by asking about what 'taller' means. It is clear that this is central to the concept of height so I will not spend more time discussing it.

The only thing left to ask now, is whether there is another way of distinguishing between the two bookcases that does not rely on a measurement or the supposedly non-existant Number property. The answer seems to be no because we can easily define the bookcases as having all their physical properties the same. If someone doubts that we can imagine two identical spatial entities in different

[27]This definition has, of course, been revisited numerous times but the point still stands.

space positions. Hence, their position will be the only way to distinguish them which as was shown is still dependent on numbers.

In this example there are several different attempted answers that need to be considered. If the only difference is in position it is crucial to realise that any argument that relies on the spatial word will have to be somewhat related to position and hence to numbers. One may attempt to avoid this by saying that one of the entities is to the left in terms of the other[28] which makes it a relationship and not simply a property. However, a relationship would imply a distinction between the two entities which makes the logic behind such an argument circular.

Another interesting approach would be to claim a distinctive non-numerical property based on non-spatial entities. The immediate example is to say that one of the entities is preferred by the person answering whether there are two different entities. However, the question then is how was this preference made. If it was arbitrary then the distinction is unreliable. If it was not then a prior distinction influenced it. Hence, any agreed upon non-spatial and even subjective argument about spatial entities must be grounded in an objective reason for which it was shown numbers are needed.

Now that we have shown that numbers are necessary to distinguish between things we have a reason to believe there are properties. What follows is an account of all the different kinds of number properties we need as well as what class of properties numbers are.

3.3.2 Different properties regarding numbers

It is important to have a clear idea of what we are talking about when referring to a 'number property'. First, we return to the Number property - a type such that all individual numbers are its tokens. The "songs of the *Iliad*", for example, possesses the Number property type and the token 24. However, it does not fall under the Number property as it is not a number and instead falls under 24. It is now clear that only numbers fall under the Number property. This seems to be a problem as it is strange for an entity to have a property type under which it does not fall. The solution to this is as follows: entities like "the songs of the *Iliad*" do not have the Number property type, instead they have number property *types*. These types, written with a non-capital letter, are the property types of which individual number. The structure of our example entity will be that it falls under the number property type 24 which falls under the Number property type.

One may then ask why it is needed that an entity such as this have a token

[28] It would have been quite unreasonable to believe in the existence of absolute directions because then if someone is asked what makes something more left than another object the answer will again involve comparison, hence numbers.

from more than one number property type. After all, we should be careful not to claim that "the songs of the *Iliad*" has more than one number as Frege pointed. However, this is not our point. It is crucial to note again the difference between properties and sets - properties must have at least two tokens to serve their purpose which is to distinguish between entities. Hence, we conclude that number property types are binary (in terms of their cardinality or number of tokens). For example, number 24 property type contains the tokens 24 and not 24. Hence, every entity possesses each number property yet has only one positive token (like '24') and infinitely many negative ones (like 'not 24').

It is useful to point that this does not cause a problem for the logicist program as it arithmetic it is equivalent to say that the range of answers is a single number and all numbers but any except a single one of them. When thinking about how an arithmetic system may work under the properties view we can consider that every number property token is not necessarily a number but a domain of values.

Please also note that a number property type cannot be physical as we explained that we cannot form a perception out of it (even a partial one) because a number means nothing without a unit. This does not have to undermine our previous point though, because we claimed that a unit is *dependent* on a number and not a number itself.

It is also interesting that there is one more property type related to numbers - the "has a number" property type. It is a binary property with tokens "has a number" and "does not have a number". This property type may not exist if the restricted domains hypothesis turns out correct and won't be a subject of that much discussion in this paper.

Now, we have provided a positive account with enough reasoning as to why numbers should be considered properties and therefore concepts. However, to convince the reader that this consideration has the potential to serve the logicist cause it will be good to at least achieve the same result as Frege's system - Frege's theorem. In Chapter 4, we discuss Hume's principle and why it is still valid under our system so as to claim the spoils of this theorem.

4 A consideration of Hume's principle

In the previous sections of this paper it was argued that numbers should be considered properties instead of objects under the Fregean system outlined in the *Grundlagen* in order to revive the logicism movement by eventually overcoming Basic Law V. Frege's own arguments against this view were considered and refuted and the metaphysical system of objects/concepts plus the *falling under* mechanism was shown to be sufficient to distinct between different entities including properties in a logical manner.

In this section, it will be established a way of incorporating Hume's principle in our metaphysical consideration of numbers because it has already been proven that the axioms of second-order Peano arithmetic can be derived by second-order logic and Hume's principle (Frege's theorem)[1]. Such a result shall hopefully be convincing in that the view that numbers are properties is not only probable but also at least equally useful as its predecessor.

4.1 Hume's principle

The principle was first outlined in Hume's *A Treatise of Human Nature* and was referenced by Frege in §73 of the *Grundlagen*[3]. After considering it and posing the Julius Caesar problem, Frege abandoned it and went on to base arithmetic on his basic laws instead. The principle states that "The number of Gs is equal to the number of Fs if and only if there exists a bijection between G and F."

On a more historic note, it may be useful to point that Hume's principle is more likely to work in a system where numbers are concepts rather than objects as the author used it in the Ancient Greek (like Aristotle's *Metaphysics* or Euclid's *Elements*) sense in which numbers are considered finite pluralities rather than Fregean objects[29]. While it is not established whether pluralities are concepts it is obvious that they are different from singular terms and, therefore, different from objects in the sense which Frege used them.

In the last chapter, we arrived at the following definition that guides the usage of properties: If α falls under a and β falls under $\neg a$ and a and $\neg a$ fall under A then α is different than β. Numbers in this definition are of the form a and $\neg a$ or α and β. While our definition provides criteria for how to distinguish between numbers and other entities, Hume's principle provides the criteria for when two entities have the same number property. A proof that our system is compatible with Hume's principle will hence consists of a way to show that the two criteria are one and the same.

Let's consider an entity F with number property α and let's say that α is 5. What this means is that F falls under $\neg 0, \neg 1, \neg 2, \neg 3, \neg 4, \neg 6, \neg 7, \neg 8, ...$ and under 5. Now, we also have some resembling case for an entity G with number property β. Now, let's say that we want to inquire about whether the two properties of the two entities are the same or not. We begin by using our definition. First, identify the binary type A in this case called *Fiveness* with tokens *is five* and *is not five*. Next, define α as falling under *is five* and conclude that "if β falls under *is not five* then $\alpha \neq \beta$."

According to Hume's principle this can happen if and only if there is not a

[29]Established in *Metaphysics* 1020a14 and *Elements* Book VII, Definitions 1 and 2 as discussed in a 2000 book by John P. Mayberry called *The Foundations of Mathematics in the Theory of Sets*.

one-to-one correspondence between F and G. Let us observe that for the concept F and its property $\neg 4$ we can identify the binary type *Fourness* and claim that if we denote this property by γ then we can define it such that γ falls under *is not four*. Hence, we are able to use our method for both "positive" and "negative" number properties.

The next step is to determine how Hume's usage of numbers fits into our system. Here, we are somewhat constrained because under our view each entity in possession of one number property possesses an infinite amount of them yet we cannot claim that the principle works for every single one of them because the cards in a standard deck and the suits in a standard deck does not possess a one-to-one correspondence yet both fall under, for example, $\neg 5$. Hence, we are forced to claim that only the following revision of Hume's principle works in our system: "The positive number property of F is equal to the positive number property of G if and only if there exists a bijection between F and G."

This can easily be shown. First, let us note that every entity that falls under the number properties falls under one and only one positive number property. This follows that for countable things it is the same to claim that such entities can fall under all negative number properties and to still be absrcibed a number by means of counting them because the intuitive number property is the positive number property. If we consider that it is possible for an entity like this to exist where the system determining whether it falls under the positive or the negative number properties for every different number is not counting but one to one correspondence[30] we are met with other difficulties. Namely, suppose such an entity exists. Then we have to ask ourselves what exactly it falling under a certain mystical big number will entail. Frege proved in an attempt to refute the psychologists' view that big numbers such that no one has thought of them exist independently and objectively. However, because they are defined by a finite number of operations we can assign things to fall under them and hence, prove they exist. But because we do not know anything about the mystical number under which F falls, we will have to conclude that we can never check if it indeed falls under it as we cannot find an entity with other known properties that fall under it except by definition defined such ones which does not serve our purpose. This establishes that every entity falls under exactly one positive number property.

Let us now establish an algorithm of how we will assign the positive number property token under which the entity F will fall such that it is compatible with

[30]This process will constitute something like this. Attempt to assign a property token for the number n such that F falls under the positive token of the n binary type by checking whether a one-to-one correspondence with another entity G defined in such a way that it falls under the said property token exists. Now, define F to be such that starting from 0 and continuing every time with $n + 1$ when a bijection does not exist one will have to perform an infinite number of operations before success. Hence, one can possibly claim that this is more controversial than counting although it will be shown that it still does not work.

Hume's principle. Let us say that we want to find that the "positive number of F" is n. We will use the one-to-one correspondence procedure. First, define the concept G such that G falls under $h = 0$. Second, proceed by checking whether a one-to-one correspondence between F and G exists. If yes, assign F to fall under 0 and infinitely many times under $\neg k$ such that $k > h$ (mitigation). If no, assign F to fall under $\neg 0$ and increase h by 1. Repeat the process until guaranteed mitigation.

Now, let us return to the example with the cards in a standard deck (F). We will repeat the procedure until on the fifty-third attempt we mitigate and will have F falling under $\neg 0, \neg 1, \neg 2, ..., \neg 51, \neg 53, ...$ and under 52. Because the same process will have to happen with say the stones in a pile of 52 stones and the result will be the same the principle holds. If something happens to the cards in the deck (say one of them is removed, for example) the result will be a new entity F_{new} for which the process can be repeated and by means of one-to-one correspondence we will still achieve the correct result (in this case distinction) if we were to compare it to the same stones of the pile.

Using this set of definitions and process we have essentially defined our algorithm to automatically uphold one of the sides in the "if and only if" part of the principle and have shown that the other works as well. This proves that there exist such as assigning algorithm that uses our metaphysical system and still results in the revised Hume's principle being an equivalent method of distinction between entities limited to positive number property tokens.

4.2 An objection regarding singular terms

Now that we have showed Hume's principle can be used to compare entities by examining positive number properties which always happens in a finite (although unknown) amount of time, we have to consider whether these entities can be numbers.

If we return to the original form of Hume's principle we can observe that numbers are referred to by a singular term and more precisely "the number of Fs" and "the number of Gs". This may serve us grounds for the skeptic to object that Hume's principle only works in the case that numbers are objects.

The answer to such an accusation comes from the significant difference between our system and Frege's - the latter distinguishes between objects and concepts based on how they are referred to in a sentence (a semantic distinction) while our system distinguishes between the two types of entities based on whether things can *fall under* them (concepts) or not (objects) making it a metaphysical distinction. In addition, notice that unlike Frege's our criteria does not grant a way to immediately tell if an entity is a concept or an object unless we prove that something falls under "able to have entities *fall under* itself" property. This, however, is not a concern because we have showed that this is the case for numbers and if one is not satisfied it is also worth mentioning that Frege's system is not purely semantic as well given that he himself notes that intuitively we used numbers as part of predicates (in §47 and §48)[3].

In order to convince ourselves that the difference between the two systems is enough we have one significant job to accomplish - establish that we can use singular terms to refer to a concept, more specifically, a property[31]. First, let us note how do we claim that objects and concepts are referred to in a sentence. For objects this happens solely by the usage of a singular term and the definite article, whereas for concepts both a singular term and the indefinite article (i.e like a predicate) can work.

This is easy to prove for objects as we agree with Frege that if the definite article is used a single entity is known. However, for concepts and hence properties the claim may require a bit more justification. First, let us give an example: "The ball is solid". Here, if we rely on our system we cannot be sure whether there is an object at all. Still, this is not a problem because both we and Frege agree that concepts can fall under concepts so the ball in this example does not matter to the purposes of the argument. Now, let's turn the attention to the predicate "is solid". We know that it refers to a concept as it is not a singular term. The question is what happens if we rephrase the example like this: "The matter state of the ball is *solid*" or even "Solid(ity) is the matter state of the ball". For the first sentence we have arrived to an equivalent structure as in Hume's principle "The number of Fs is..." Now, according to Frege the first example and this sentence does not refer to the same concept. But we claim that he is wrong because of the usage of "of the ball" which signifies a property which under both systems has the ball falling under it. Our last example makes things even clearer. Let's assume we have agreed on the existence of a property of the ball like state of matter. How are we then to inquire semantically about this property? Our last sentence accomplishes this by stating that "Solid(ity) is the state of" Which mentions a token (solid(ity)) and a type (state) but as we have established in a previous chapter the ball actually falls under the token because it is not a state of matter but a solid entity[32].

The observation that entities fall under what is often thought as a token and not a type combined with the necessity of the type in order to use properties for their established purpose makes it clear that if we are to distinguish entities analytically (which will imply the need of semantics) we will have to admit that token properties (which can also be types) can be referred to by the usage of a singular term.

If this is not the case we will never be able to distinguish between concepts

[31]We are only concerned with properties because under our systems numbers are always properties and not other kinds of concepts.

[32]In addition, because solidity is a *physical* property from here it will be easy to prove that the ball is indeed an object.

as concepts words does not give an accurate description of a concept, instead providing an entity that falls under it. This means that in essence our claims and examples are meant to challenge one of Frege's principles that one can not use a singular term to refer to a concept without altering it. While, the complete truth here remains a subject to debate, it is certain that if one permits the distinction between concepts based on the *falling under* mechanism that we have proved works, one has to admit that even such an alternation or new entity keeps an equivalent relationship with the original concept in terms of what the concept is and is not. This means that no matter what are we referring to by using "the number of Fs" and "the number of Gs" in Hume's principle these still allow us to make claims about what the number property is and is not in sentences like "The Fs are (a number)." and "The Gs are (a number)."

We now have only one last thing to discuss - how are singular terms and predicates defined in our system. This is important because without connecting the metaphysical reality about objects and concepts to the semantics we will never be able to prove that we are still consistent with Hume's principle.

4.2.1 Relationship between concepts, objects and predicates, singular terms

Frege's system has the advantage over the one devised in this paper of a very clear connection between the metaphysical reality and the semantic entities. In the original system singular terms always designate objects and predicates always designate concepts. This allows Frege to create clear definitions for the semantic entities. He can claim that singular terms are the entities that designate objects and predicates are what is left in a sentence one all the singular terms are removed.

However, such a definition is, in fact, providing the opportunity for a nice way out for the current system. While it is allowed for singular terms to designate both objects and concept, the same is not true for predicates which can designate concepts and only concepts. This allows us to define predicates as the semantic entities designating concepts exclusively and singular terms as the semantic entities that are left once we remove the predicates from a sentence[33].

One may notice that this system directly eliminates the concept horse paradox given that there is no longer a problem with a sentence such as "The concept horse is an object." or "The concept horse is a concept." with both being able to be assigned truth values given that singular terms can designate both concepts and objects.

Hence, we have showed how an equivalent revision of Hume's principle can

[33]Here, it is vital to note the word 'exclusively' in the definition of predicates as simply saying that they are entities which designate concepts is not enough given that singular terms can do that just as well.

be used in our system to distinguish the number properties of different entities without suffering from the first order - second order difference depending on what numbers are. Since the mechanism stays largely the same, we still benefit from the result of Frege's theorem.

5 Conclusions

The main point of this paper was to serve as a guidelines for new research into logicism based not on neo-logicism and the Fregean system but a new and revised system under which numbers are properties and by extension - concepts. The motivation behind logicism is clear a reduction of both the metaphysics and epistemology needed to explain mathematics to that needed to explain logic will be of great importance given that because mathematics is reliant on logic, the latter will still have to be explained no matter our views about mathematics.

In Chapter 2, the metaphysical system Frege employed from his paper *On Concept and Object* throughout the *Grundlagen* was explained and discussed. It was shown that the difference between concepts and objects is semantic and that only the former can have entities fall under them. It was argued against the first view and in favour of the second. Moreover, *falling under* was identified as the exclusive relationship upon which to distinguish metaphysically between objects and concepts. This led to a new semantic view about the metaphysical entities in which singular terms are no longer constrained to designating objects. In addition, the nature of properties was discussed. In agreement with Frege, it was concluded that properties are concepts and employing Oliver's type and token terminology options were explored about the *domain* of property types. In addition, the terminology *cardinality* of a property type was introduced in the same way as cardinality of a set and it was argued that the distinction between property types and sets lies in the fact that the cardinality of the former having to be at least 2 whereas no such constrains are placed on the latter. Property types with cardinality of exactly 2 were named binary. Lastly, a difference was made between *physical* and *non-physical* properties in terms of the former inducing a partial sensual perception which implies that the entities that fall under them are *spatial*.

In the first part of Chapter 3, Frege's arguments against Mill's view that numbers are *physical* properties were discussed in three groups - about numbers in general, about specific numbers, and about *physical* properties. It was shown that there are not sufficient to convince one that numbers are not properties. In the second part of Chapter 3, the inheritance structure of the *falling under* relationship was explored culminating in a view that only binary types can be used to distinguish between entities. A logical definition was presented in the following form: "x is different than y if and only if x falls under a, y falls under $\neg a$, and both a and $\neg a$ fall under A where A is a binary property type. In the last part of this chapter, it was shown that it is not possible to distinguish between *spatial* entities without admitting that numbers are properties even if other properties like *colour* or *solidity* seem to be enough. This was the essence of the positive account of the view that numbers are properties aimed at convincing the reader of that same view once Frege's arguments have been refuted.

In Chapter 4, the attention was turned to Hume's principle so as to show that the new system from this paper is consistent with it and can still enjoy the benefits of Frege's theorem. In the first part, an algorithm was provided about understanding whether two properties are equinumerous in a finite amount of steps even when the exact amount is unknown. In the latter parts, an objection about the fact that Hume's principle relied on singular terms was discussed and refuted on the grounds that in the new system they can designate concepts as well as objects. Last, in order to convince the reader that there are strict meanings behind singular terms and predicates, the latter were defined as semantic entities designating concepts exclusively, and the former as the semantic entities left after all the predicates were removed from a sentence. As a side effect, it was observed that this system eliminates the concept horse paradox.

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