First-Order Models of Vector Spaces over $\mathbb{Q}$ and $\mathbb{F}$

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Abstract

We give a language and a set Σ of sentences of that language such that any structure $\mathfrak{A}$ that is a model of Σ is a vector space over $\mathbb{Q}$. We then give sets Γ_n for each n such that the vector space given by $\mathfrak{A}$ is of dimension n or more exactly when $\mathfrak{A}$ satisfies Γ_n with some variable assignment. Then we show that $Cn(\Sigma)$ is incomplete and discuss what extensions make it complete. Throughout the paper we consider how our conclusions would differ if the vector spaces we were considering were over a finite field $\mathbb{F}$ instead of $\mathbb{Q}$. Finally, we discuss the implications of these results for determining the dimension of a vector space given by such a structure $\mathfrak{A}$.

1 Introduction

Vector spaces are sets equipped with addition and scalar multiplication (with members of a field). They constitute an interesting algebraic structure that can be modeled in a countable first order language when the vector spaces we want to model are over a countable field. In this paper, we give a language $\mathcal{L}$ for this purpose. Then we give an infinite, though countable, set of sentences Σ of $\mathcal{L}$ such that a structure $\mathfrak{A}$ is a vector space over the field of rational numbers $\mathbb{Q}$ when $\mathfrak{A}$ is a model of Σ. We will also give a revised language $\mathcal{L}_{\mathbb{F}}$ and a set $\Sigma_{\mathbb{F}}$ such that any structure of $\mathcal{L}_{\mathbb{F}}$ that satisfies $\Sigma_{\mathbb{F}}$ is a vector space over $\mathbb{F}$ where $\mathbb{F}$ is an arbitrary finite field.

We then proceed to show that there exists a set of formulas Γ_n such that the vector space given by $\mathfrak{A}$ is of dimension at least n exactly when $\mathfrak{A}$ satisfies Γ_n with some variable assignment. We will explain why we cannot guarantee a vector space will have a dimension of exactly n. Then will also give a set of sentences Δ_n such that any structure $\mathfrak{A}_{\mathbb{F}}$ of $\mathcal{L}_{\mathbb{F}}$ that is a model of $\Sigma_{\mathbb{F}}$ and that satisfies Δ_n is a vector space of dimension exactly n. The primary motivation behind this is to show that our language $\mathcal{L}$ and set Σ provide for a useful and not just correct description of vector spaces over $\mathbb{Q}$. The same also applies for $\mathcal{L}_{\mathbb{F}}$ and $\Sigma_{\mathbb{F}}$.

We proceed to establish that the consequences of Σ do not constitute a complete theory by giving an example sentence such that neither it nor its negation have a deduction from Σ. Then it is shown that the union of Σ with this sentence form a complete theory and discuss whether they are other ways of extending Σ to achieve a complete theory.

In the rest of this section we define $\mathcal{L}$ and Σ as well as $\mathcal{L}_{\mathbb{F}}$ and $\Sigma_{\mathbb{F}}$. Then in section 2, we prove their utility by showing that they allow us to form sets of sentences such that we have a vector space of at least a given dimension exactly when a model $\mathfrak{A}$ of Σ satisfies the respective set with some variable assignment. We manage to give a set of sentences that will guarantee a vector space over a finite field satisfying the sentences in the set is exactly of a given dimension. In section 3, we show that $Cn(\Sigma)$ is incomplete and that we can extend it to a complete theory by adding a single sentence. We explore what happens when we try to extend it with different sentences. Last, in section 4, we summarize and discuss the results.

1.1 The Language $\mathcal{L}$ for Vector Spaces over $\mathbb{Q}$

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Let $\mathcal{L}$ be a first-order language with equality, the constant symbol 0, the two place function symbol + and the members of the set $\{s_q \mid q \in \mathbb{Q}\}$ where each s_q is a one place function symbol. The constant symbol 0 will be used to represent the zero vector which is present in every vector field while the + function symbol will be used for the vector addition operation. The s_q function symbols are going to represent scalar multiplication of the vector they are applied to by a specific scalar q.

We note some observations about $\mathcal{L}$. We can divide all symbols in $\mathcal{L}$ in three groups such that group 1 consists of the variable symbols, group 2 consists of all the one place predicate functions, and group 3 consists of everything else. First, $\mathcal{L}$ is infinite because we have infinitely many variable symbols in it. Second, $\mathcal{L}$ is countable. To show this we note that group 1 is countable by definition, group 2 is finite, and whether group 3 is countable depends on whether the set $\{s_q \mid q \in \mathbb{Q}\}$ is countable. Its cardinality is equal to the cardinality of $\mathbb{Q}$ and the cardinality of $\mathbb{Q}$ is equal to the cardinality of $\mathbb{N}$. Hence group 3 is countable and so is $\mathcal{L}$.

In addition, note that $\mathcal{L}$ is not the minimal language we could have used. First, it features both the universal and existential quantifier for the sake of simplicity when listing the axioms of vector spaces in $\mathcal{L}$ and we could have accomplished this task without the existential quantifier be it in a longer way. As it turns out the same applies for the constant symbol 0 which is included with the purpose of representing the additive identity which appears in the axioms for vector spaces and is also present in any vector space. When we give the set Σ we will show that there is a way to give the axioms involving the identity without a constant symbol, but we will continue to use it as it does not complicate any of the proofs in this paper. In addition to using the existential quantifier as a simplification, we would also allow the usage of $x = y$ instead of $= xy$ and similarly the usage of $x + y$ instead of $+xy$.

1.2 The Set of Axioms for Vector Spaces Σ in $\mathcal{L}$

To list the axioms for vector spaces in our first-order language, we will use the definition of vector spaces given by Friedberg, Insel, and Spence in their fifth edition of their linear algebra textbook[2]. A vector space V is defined as a set equipped with two operations over a field $\mathbb{F}$ such that for any $x, y \in V$ and $a \in \mathbb{F}$ there exists a $z \in V$ such that $x + y = z$ and there exists a unique $w \in V$ such that $w = ax$ and the following conditions hold:

(VS 1) For all $x, y \in V$, $x + y = y + x$.
(VS 2) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$.
(VS 3) There exists an element $0 \in V$ such that for all $x \in V$, $x + 0 = x$.
(VS 4) For each $x \in V$, there exists a $y \in V$ such that $x + y = 0$.
(VS 5) For each $x \in V$, $1x = x$.
(VS 6) For each $a, b \in \mathbb{F}$ and for each $x \in V$, $(ab)x = a(bx)$.
(VS 7) For each $a \in \mathbb{F}$ and for each $x, y \in V$, $a(x + y) = ax + ay$.
(VS 8) For each $a, b \in \mathbb{F}$ and for each $x \in V$, $(a + b)x = ax + bx$.

We will separate our sentences in eight sets denoted by Ai where $i \in [1; 8]$. We have. Sets A1-A5 consist of a single element whereas sets A6-A8 are countably infinite. We have:

(A1) $\forall x \forall y((x + y) = (y + x))$
(A2) $\forall x \forall y \forall z(((x + y) + z) = (x + (y + z)))$
(A3) $\forall x((x + 0) = x)$
(A4) $\forall x \exists y((x + y) = 0)$
(A5) $\forall x(s_1 x = x)$

To satisfy (VS 6 - VS 8) we will need an infinite number of sentences for each given that there are infinitely many one place function symbols in the language that serve for multiplying a variable by a specific rational number. Hence, when I give the sentences for (A6-A8) it may appear as a single sentence but it is in fact infinitely many sentences such that there is one for each $a, b \in \mathbb{Q}$.

(A6) $\forall x((s_{ab}x) = (s_a(s_b x)))$
(A7) $\forall x \forall y((s_a(x+y)) = ((s_a x) + (s_a y)))$
(A8) $\forall x((s_{a+b}x) = ((s_a x) + (s_b x)))$

There are several things to note before proceeding. First, note that at places we are treating $\mathbb{Q}$ not simply as a set, but as a field. This means that for any $a, b \in \mathbb{Q}$ we are guaranteed that there exist $ab \in \mathbb{Q}$ and $a + b \in \mathbb{Q}$. That is why we are allowing ourselves to write s_{ab} and s_{a+b} in the shorthand notation for describing the form of the infinite sentences needed to satisfy (VS 6) and (VS 8) respectively. Second, observe that technically we are not required to have the constant symbol 0 in our language. We will continue to use it for simplicity but to show how we can satisfy (VS 3) and (VS 4) without it I put forward:

(A3*) $\exists z(\forall x((x+z) = x))$
(A4*) $\exists z(\forall x(((x+z) = x) \wedge \exists y((x+y = z))))$

Now that we have made these remarks, we let Σ be the union of sets A1-A8. Then any structure $\mathfrak{A}$ will be a vector space over the rationals exactly when $\mathfrak{A}$ satisfies Σ.

We now proceed to define a language and give a similar set for vector spaces over an arbitrary finite field $\mathbb{F}$.

1.3 The Language $\mathcal{L}_{\mathbb{F}}$ for Vector Spaces over $\mathbb{F}$

The language $\mathcal{L}_{\mathbb{F}}$ again contains an infinite number of variables and the same connectives as $\mathcal{L}$. We define it in this section. Let $\mathcal{L}_{\mathbb{F}}$ be a revised language of $\mathcal{L}$, which does not include the function symbols of the form s_q but includes finitely many one-place function symbols f_a for $a \in \mathbb{F}$.

We will also let $\mathbb{F}$ be such that $\mathbb{F}$ has exactly k elements in it. Note that in addition to being natural, k must be equal to a number p^m where m is a natural number and p is a prime number[4]. This is necessary so that it is possible that $\mathbb{F}$ is a field in the first place and not another kind of an algebraic structure such as a ring.

We can now use this language $\mathcal{L}_{\mathfrak{F}}$ to give a set of sentences $\Sigma_{\mathbb{F}}$ that when satisfied by a structure $\mathfrak{A}_{\mathbb{F}}$ will ensure it is a vector space over $\mathbb{F}$.

1.4 The Set of Axioms for Vector Spaces $\Sigma_{\mathbb{F}}$ in $\mathcal{L}_{\mathbb{F}}$

Let $\Sigma_{\mathbb{F}}$ be our revised set of axioms so that instead of groups (A6-A8) we have groups (A6*-A8*) such that for each $a, b \in [1; k]$:

(A6*) $\forall x((f_{ab}x) = (f_a(f_b x)))$
(A7*) $\forall x \forall y((f_a(x + y)) = ((f_a x) + (f_a y)))$
(A8*) $\forall x((f_{a+b}x) = ((f_a x) + (f_b x)))$

Since only the scalar multiplication function symbols are different in the language for vector spaces over $\mathbb{F}$ we do not need to change anything about the sentences in axiom groups (A1-A5). The same observations that we made when detailing Σ continue to be true. In particular, the constant symbol 0 is again unnecessary but we employ it for simplicity. The same shorthand conventions apply as well.

It again follows, that any structure $\mathfrak{A}_{\mathbb{F}}$ of $\mathcal{L}_{\mathbb{F}}$ that satisfies $\Sigma_{\mathbb{F}}$ is a vector space yet this time it is over the finite field $\mathbb{F}$ and not over $\mathbb{Q}$.

2 Dimensions of Vector Spaces Given by $\mathfrak{A}$

In this section, we show that 1) for any given natural n a vector space $\mathfrak{A}$ (being a model of Σ) has dimension of at least n exactly when it satisfies a set Γ_n with some variable assignment s and that 2) there is a set Δ_n such that $\mathfrak{A}_{\mathbb{F}}$ has dimension exactly n exactly when it satisfies Δ_n.

We begin with some definitions from Friedberg's book. We define the dimension of a vector space V as $dim(V)$ such that $dim(V)$ is equal to the number of vectors in a basis for V and as a special case $dim(\{0\}) = 0$. We define a basis for V as a set of linearly independent vectors that span V. A finite set of vectors is said to be linearly independent if no vector in the set can be written as a linear combination of other vectors in the set. An infinite set of vectors is said to be linearly independent if all of its finite subsets are linearly independent. The span of a set of vectors is defined as the set of all vectors that can be written as a linear combination of the vectors in the set. Last, a linear combination of a set of vectors is defined as the vector obtained by multiplying every vector in the set by some element of the field over which the vector space is (not necessarily the same elements of the field) and adding them together[2].

It is a known result that every basis for a vector space has the same number of elements and that if a vector space V has dimension n then any linearly independent set of n vectors is a basis for V[2]. Now note that our definition of linear independence for a set is equivalent to the sum of each vector in the set multiplied by a scalar from the field equalling 0 exactly when every chosen scalar is 0.

The idea in this section is as follows. First, we give a set of formulas Γ_n such that any model of Σ that satisfies Γ_n (with some variable assignment) has dimension at least n. We also explain why we need free variables to occur in Γ_n. After that we proceed with giving a set of sentences Δ_n which when satisfied by a structure $\mathfrak{A}_{\mathbb{F}}$ will ensure it is of dimension exactly equal to n. Let us proceed with the case for vector fields over $\mathbb{Q}$.

2.1 The Sets Γ_n for Vector Spaces of Dimension at Least n

Let us begin with some easy cases for n. When $n = 0$ we know that the vector space is exactly $\{0\}$. Then Γ_0 is empty since every vector space contains 0. If $n = 1$ we basically need to ensure that the vector space is not $\{0\}$. We set $\Gamma_1 = \{\exists x(\neg(x = 0))\}$.

It gets more interesting when $n \geq 2$. At this point it becomes apparent that Γ_n cannot be a set of sentences for $n \geq 2$ and has to be a set of formulas instead. The idea behind this is that if Γ_n was to be a set of sentences we would want to have only bound variables in our sentences. At the same time, however, we will have to ensure that our set guarantees that there exist at least n linearly independent vectors in any vector space $\mathfrak{A}$ that satisfies Γ_n. However, even in the simplest case when $n = 2$ we would need infinite checks to ensure that there exist two vectors x and y such that $x \neq qy$ for any $q \in \mathbb{Q}$. But we cannot quantify over the scalars since and we represent them by using function symbols so this procedure would require us to use an infinite sentence no matter how many sentences are included in Γ_n. If we bound x and y by an existential quantifier we will need one check for each function symbol in $\mathcal{L}$ in this single sentence. This, however, would make it infinite and all sentences are finite. We could also not get around this issue by employing the universal quantifier and requiring that any two vectors in a vector space that are non-zero are linearly independent. One supposed way to do this would be to have the sentence

$$\exists x \exists y(\neg(x = y) \wedge \neg(x = 0) \wedge \neg(y = 0))$$

and the sentences

$$\forall x \forall y((\neg(x = y) \wedge \neg(x = 0) \wedge \neg(y = 0)) \rightarrow (\neg(s_q x = y)))$$

where we have one such sentence for every s_q in our language. Now, let Γ_2 be the set of all such sentences and the sentence that guarantees the existence of two different non-zero vectors. However, the latter group of sentences will always be false because for every non-zero vector space $\mathfrak{B}$ there is a non-zero element $x \in \mathfrak{B}$ and all $s_q x$ are in $\mathfrak{B}$ while they are not linearly independent with x. Hence, it is not possible for Γ_n to be a set of sentences when $n \geq 2$. It needs to be a set of formulas instead.

Let us start with our simple case $n = 2$ and see what Γ_2 might be. Essentially we want to have free variables v_1 and v_2 such that for no s_q it is true that $v_1 = s_q v_2$. Then let Γ_2 be the following set:

$$\Gamma_2 = \{\neg(v_1 = s_q v_2) \,|q \in \mathbb{Q}\}$$

If there does not exist a variable assignment s such that a vector space $\mathfrak{A}$ with s satisfies Γ_2 then for any two elements of $\mathfrak{A}$ there exists a scalar such that one of them equals the respective function symbol applied to the other. Then there are no 2 linearly independent vectors and hence $\mathfrak{A}$ is of dimension 0 or 1. If such a variable assignment exists, then there is a set of cardinality 2 that is linearly independent and $dim(\mathfrak{A}) \geq 2$.

Next, we generalize this for any $n \geq 2$. We use some shorthand notation. We define $v_1 \neq v_2$ to mean $\neg(v_1 = v_2)$. Hence, we define Γ_n as the following:

$$\Gamma_n = \{s_{i_1} v_1 + s_{i_2} v_2 + \cdots + s_{i_n} v_n \neq 0 \,|\, i_1, i_2, \ldots, i_n \in \mathbb{Q}\}$$

Then a generalization of the argument for the case $n = 2$ ensures that a model of Σ, $\mathfrak{A}$ is of dimension at least n if and only if $\mathfrak{A}$ satisfies Γ_n with some variable assignment s.

Unfortunately, we can not give a set of formulas such that a vector space $\mathfrak{A}$ satisfying this set has dimension exactly n except when $n = 0$. When that is the case we simply require the sentence

$$\forall x(x = 0)$$

However, when $n > 0$ our vector space is infinite. We will need to guarantee that there are no $n+1$ linearly independent vectors which would require using the universal quantifier to bound $n+1$ variables. But then we run with the infinite sentence problem again because since we cannot quantify over the elements of $\mathbb{Q}$ we will have to produce and infinite conjunction guaranteeing for no scalars those elements turn out to be linearly independent.

This, however, is not the case when we work out with the finite field $\mathbb{F}$. We explore how to define sets Δ_n of sentences that when satisfied will guarantee that a vector space $\mathfrak{A}_{\mathbb{F}}$ has dimension exactly n.

2.2 The Sets Δ_n for Vector Spaces of Dimension Exactly n

In this section, we give the set of sentences Δ_n such that any $\mathfrak{A}_{\mathbb{F}}$ that is a vector space (satisfies $\Sigma_{\mathbb{F}}$) and satisfies Δ_n has dimension exactly n.

Again, we begin with some trivial cases. When $n = 0$ we want 0 to be the only element of $\mathfrak{A}_{\mathbb{F}}$. Consequently, $\Delta_0 = \{\forall x(x = 0)\}$. When $n = 1$ we want to guarantee that there is a non-zero vector and that all distinct non-zero vectors are linearly dependent. Hence we have $\Delta_1 = \{\exists x(x \neq 0) \wedge \forall y(x = f_1 y \vee x = f_2 y \vee \cdots \vee x = f_k y)\}$.

When $n \geq 2$ we construct Δ_n by first having a sentence that guarantees the existence of n distinct non-zero vectors (we use $x \neq y$ for $\neg(x = y)$):

$$\exists x_1 \exists x_2 \ldots \exists x_n((x_1 \neq 0 \ldots x_n \neq 0) \wedge (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \cdots \wedge x_{n-1} \neq x_n))$$

which we will abbreviate as (A_n) and then adding the infinite many sentences of the form

$$A_n \wedge \forall x_1 \forall x_2 \ldots \forall x_n(A_n \rightarrow (f_{a_1} x_1 + f_{a_2} x_2 + \cdots + f_{a_n} x_n \neq 0))$$

where we have one such sentences for all combinations of scalar multiplication functions different from the 0 multiplication function (we denote this by f_0) but it is actually one of the functions f_i for i between 1 and k. We can employ this formally by shifting a in the definition of f_a to be between 0 and $k-1$ and then requiring the sentence

$$\forall x(f_0 x = 0)$$

Now, the union of these sentences will guarantee that a vector space $\mathfrak{A}_{\mathbb{F}}$ that satisfies the union has dimension at least n. To guarantee that the dimension is in fact n we need to add a sentence that says:

$$\forall x_1 \dots \forall x_{n+1}(A_{n+1} \to (F_1 \vee F_2 \vee \cdots \vee F_{{k_1}^{n+1}}))$$

where $F_1 = (f_1x_1 + f_1x_2 + \cdots + f_1x_{n+1} = 0)$, $F_2 = (f_1x_1 + f_1x_2 + \cdots + f_2x_{n+1} = 0)$, and so on with $F_{{k-1}^{n+1}} = (f_{k-1}x_1 + f_{k-1}x_2 + \cdots + f_{k-1}x_{n+1} = 0)$. This sentence makes sure that for any distinct non-zero $n+1$ elements in $\mathfrak{A}_{\mathbb{F}}$ there will always be non-zero scalars proving that these $n+1$ elements are linearly dependent. And if that is indeed the case for any $n+1$ distinct non-zero elements then the dimension of $\mathfrak{A}_{\mathbb{F}}$ is less than $n+1$.

Finally, the union of this sentence with the sentences ensuring dimension of at least n guarantees dimension of exactly n. This union is then Δ_n.

Having shown some of the things we can accomplish in our model for vector spaces we now turn the attention to whether the consequences of our axioms constitute a complete theory.

3 Incompleteness of $Cn(\Sigma)$ and Its Complete Extensions

In this section, we will show that $Cn(\Sigma)$ is incomplete. We will give an example of a sentence such that neither it nor its negation have a deduction from Σ. Then we will show that by adding this sentence to Σ we do end up with a complete theory.

We begin by introducing the notion of categoricity. We use the definition given by Enderton in his *A Mathematical Introduction to Logic*. We say that 'given a cardinal κ and a theory T, "T is κ-categorical iff all models of T having cardinality κ are isomorphic"[1].

From Enderton we also introduce the Los-Vaught Test:

Let T be a theory in a countable language. Assume that T has no finite models. (a) If T is $\aleph_0$-categorical, then T is complete. (b) If T is κ-categorical for some infinite cardinal κ, then T is complete.[1].

We will use these to later prove that our extensions of Σ are complete. Specifically, we will employ part (b) of the Los-Vaught Test. First, however, we show that $Cn(\Sigma)$ is incomplete.

3.1 A Sentence σ Such That Σ Does not Prove or Disprove σ

Consider the following sentence σ:

$$\exists x(x \neq 0)$$

This sentence asserts that there exists an element in any $\mathfrak{A}$ satisfying it such that it is not 0. This results in $\mathfrak{A}$ (when also satisfying Σ) being a non-zero vector space over $\mathbb{Q}$. Let us first, consider whether $\Sigma \vdash \sigma$. This is not the case because there exists a structure $\mathfrak{A}$ such that $\mathfrak{A}$ satisfies Σ and $\mathfrak{A}$ satisfies Γ_1 meaning that $dim(\mathfrak{A}) \geq 1$ and $dim(\mathfrak{A}) < 1$ iff $\mathfrak{A} = \{0\}$. One example of such a structure would be the real numbers over the rationals. Now, let us consider whether $\Sigma \vdash \neg\sigma$. This is again not the case since the structure $\mathfrak{A} = \{0\}$ satisfies both Σ and σ. From this we can conclude that it is not the case that $\Sigma \vdash \sigma$ or that $\Sigma \vdash \neg\sigma$. But then by the existence of σ, $Cn(\Sigma)$ is incomplete.

We now proceed to argue that the theories of $\Sigma \cup \{\sigma\}$ and $\Sigma \cup \{\neg\sigma\}$ are complete.

3.2 On the Completeness of $\Sigma \cup \{\sigma\}$ and $\Sigma \cup \{\neg\sigma\}$

Next, we want to show that the theory of the extension of Σ with σ or $\neg\sigma$ are going to be complete. We will do this by applying the Los-Vaught test. We show that if κ is the cardinality of the real numbers, then $\mathbb{Q}$ is κ-categorical. We will want to show that the theory of non-zero vector spaces over $\mathbb{Q}$ is categorical in the cardinality of the reals. Then by the test we will conclude that the theory of non-zero vector spaces over $\mathbb{Q}$ is complete.

To show that the theory of non-zero vector spaces over $\mathbb{Q}$ is categorical in the cardinality of the reals, we need to demonstrate that any two models of this theory with the same cardinality of the reals are isomorphic.

Let V and W be two non-zero vector spaces over $\mathbb{Q}$ with the same cardinality of the reals. Since V and W are non-zero, they have a basis, and any two bases of a vector space have the same cardinality. Let B be a basis of V and let C be a basis of W, both with the same cardinality of the reals.

Since V and W have the same cardinality of the reals, there exists a bijection $h : B \to C$ between the bases. We can extend h to a linear map $g : V \to W$ by defining $g(b) = h(b)$ for all b in B and then extending linearly to all of V. That is, for any v in V, we can write v as a finite linear combination of basis elements, $v = a_1b_1 + ... + a_nb_n$, and then set $g(v) = a_1h(b_1) + ... + a_nh(b_n)$.

It remains to show that g is an isomorphism of vector spaces. To see this, note that g is a linear bijection (since it is a bijection on the basis and extends linearly). Moreover, for any q in $\mathbb{Q}$ and any v in V, we have $g(qv)$ $= qg(v)$, since g is linear. Therefore, g preserves the structure of the field $\mathbb{Q}$ acting on V. Thus, g is an isomorphism of vector spaces.

Therefore, we have shown that any two non-zero vector spaces over $\mathbb{Q}$ with the same cardinality of the reals are isomorphic, which implies that the theory of non-zero vector spaces over $\mathbb{Q}$ is categorical in the cardinality of the reals.

Then by part (b) of the Los-Vaught test we conclude that the theory of non-zero vector spaces over $\mathbb{Q}$ is complete. Now, notice that the theory of $\Sigma \cup \{\sigma\}$ refers exactly to that theory as Σ guarantees a vector space over $\mathbb{Q}$ while σ guarantees there exists a non-zero element. But then $\Sigma \cup \{\sigma\}$ must be complete. It is also easy to see that $\Sigma \cup \{\neg\sigma\}$ is complete. That is because it refers specifically to the vector space $\{0\}$ and given that we know that it contains exactly one vector it follows that the theory will be complete.

It is now worth considering whether there are any other ways to extend Σ, so that we can derive a complete theory. It can be seen that there are infinitely many sentences by which to extend Σ and have the same theory we have proven to be complete. These are of the form $\neg\neg\sigma$, $\neg\neg\neg\sigma$, $\sigma \wedge \neg\forall x(x + 0 \neq 0)$ and so on. We can also always extend Σ to an inconsistent set and therefore its theory will be complete since every inconsistent theory is complete. To this we may add a sentence τ that states $\forall x(x+0 \neq 0)$ which contradicts the sentence in (A3). This leaves us with 3 theories extending $Cn(\Sigma)$ which are complete. However, any other set with a theory that is consistent but fails to determine whether there exists a non-zero vector in any $\mathfrak{A}$ satisfying it will not be able to prove σ or $\neg\sigma$ and will hence be incomplete. Therefore, in total three theories are both extensions of $Cn(\Sigma)$ and are complete.

We now proceed to summarise our results and draw certain conclusions from them.

4 Conclusions

In this paper, we have given languages $\mathcal{L}$ and $\mathcal{L}_{\mathbb{F}}$ to be used to model vector spaces over $\mathbb{Q}$ and an arbitrary random field $\mathbb{F}$ respectively. We have also given two sets Σ and $\Sigma_{\mathbb{F}}$ such that any structures $\mathfrak{A}$ and $\mathfrak{A}_{\mathbb{F}}$ satisfying Σ and $\Sigma_{\mathbb{F}}$ respectively are vector fields over $\mathbb{Q}$ and $\mathbb{F}$ respectively.

We then proceeded to prove the utility of our languages and sets by giving sets of formulas Γ_n such that a vector space $\mathfrak{A}$ satisfying Γ_n has dimension of at least n. We have explained why it is impossible to give such a set of sentences or give a set of formulas such that we are guaranteed dimension n except when $n = 0$. We can then conclude that any vector space $\mathfrak{A}$ such that $\mathfrak{A} \models \forall x(x = 0)$ is such that $dim(\mathfrak{A}) = 0$. As mentioned we can also have a minimal bound for the dimension of any $\mathfrak{A}$ but cannot guarantee any dimension apart from 0 because there exists no sentence on which this depends. Moreover, every non-zero vector space over $\mathbb{Q}$ has an infinite number of elements. We have then observed that we can give a set of sentences for each n such that a vector space over $\mathbb{F}$, $\mathfrak{A}_{\mathbb{F}}$ that satisfies them has dimension at least n and we can extend them to the set of sentences Δ_n which when satisfied guarantees dimension exactly equal to n. Hence any $\mathfrak{A}_{\mathbb{F}}$ that satisfies Δ_n is such that $dim(\mathfrak{A}_{\mathbb{F}}) = n$.

We then continued to show that $Cn(\Sigma)$ is not a complete theory by establishing that it neither proves nor disproves the sentence σ stating that there exists a non-zero vector. We have also argued that the theory of non-zero vectors over $\mathbb{Q}$ is complete by showing they are categorical to the cardinality of the reals and applying the Los-Vaught Test. Then we concluded that the theories of $\Sigma \cup \{\sigma\}$ and of $\Sigma \cup \{\neg\sigma\}$ are complete. We have also argued that the only other complete theory extending $Cn(\Sigma)$ is the inconsistent theory. We did not argue anything about the completeness or incompleteness of $Cn(\Sigma_{\mathbb{F}})$. This problem is addressed by Richard Kaye[3] and may serve as an inspiration for future discussion.

5 References

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