Comments on *The Foundations of Arithmetic*

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Contents

1 Introduction

The document contains descriptions of some arguments presented by Frege in terms of the choices he makes when attempting to explain the nature of cardinal numbers.

Organized in subsections, major arguments are presented in terms of how I understood them, and are followed by comments and, possibly, ideas building up on them.

The arguments are divided in several sections depending on what they deal with.

2 Difference between definite and indefinite

2.1 Difference between 1 and a

Frege opens the paper with suggesting that people may answer the original question (what is the number one) by saying that "the number one is a thing". However, this is not a definition because 1 is concrete (proven by the usage of 'the') while 'thing' refers to a collection of different concretes. Hence, it will be possible to choose anything to be the meaning of 1 under this definition.

*Please excuse any spelling / grammar errors.

Frege points out that the question may be meaningless because the parameter a cannot be defined either because a statement such as $a - a = 0$ is true for whatever specific cardinal number we choose. However, he dismisses this accusation because in equations such as $1 + 1 = 2$ such substitution for 1 is not possible. Thus there is a difference between specific numbers and parameters.

Comment: It is interesting to note the implications of this "accusation" in terms of the initial proposed naive answer. Frege's objection to that answer had to do with the concreteness required by the definite article. It is natural then to observe if a naive but valid definition can be given to the parameter a can be defined in the context of say $a - a = 0$. It seems that this can be accomplished by something like "The parameter a is a number[1]".

This is, however, a bit problematic as it seems to be both valid and still using both the definite and indefinite articles. If this is true, it may follow that Frege's point may not be enough to justify his opposition to such grammatical constructs.

Let's note that if we do not require the presence of an article it is possible to say "a is a number". However, it seems logical that everything is either definite or indefinite, so that either "*The* a is a number" or "*A* a is a number" must be equivalent to the original statement. There is also the needed addition of the word 'parameter' in order to make the sentence grammatically correct. It now seems that we cannot immediately distinguish our parameter a from any general parameter (that is still a substitute of a number). This is problematic. On one hand, we have the problem about the validity of a statement that uses both the definite in the indefinite article in a structure similar to the naive definition with which Frege begins his work. On the other, we have arrived at a dilemma about parameters in general.

It is safe to say that Frege considers numbers to be objects and thus to be referred by a singular term in opposition to concepts. It is then interesting to ask what are parameters. On one hand, our attempt has shown that we are not able to precisely distinguish our parameter a from any other such forms. On the other, it is also true that we have established a property of a - it is a substitute for a *number* and not for say the *Moon* (the context remains $a - a = 0$). This raises important questions about Frege's classification system as to what can be defined, and what can have properties. Under his system there is a hierarchy of different concepts but not of objects. This may mean that he does not consider parameters to be objects given that it not be that hard to define a parameter that is a substitute for a parameter like a (such as by attempting to say that it is a substitute for a member of a collection of numbers that may either be all even, odd, bigger than 5, and so on, thus introducing another parameter[2]).

2.2 Third principle

In his introduction (close to the end), Frege states the three principles to which he will adhere in the remain of his work. The third one is about always keeping in mind the difference between an object and a concept.

[1]Like in the paper, I will refer to countable numbers exclusively

[2]Not that great since it is not needed in this situation, and some may argue that because of that such different collections serving as mediums between a and any number does not, in fact, exist.

Comment: As we discussed before, the main difference is semantic given this is the criteria for whether something is an object (namely if it can be referred to by a singular term). In addition, there is the fact that only objects combine with subjects to form statements that hold some judgment. This, I think, provides a strong argument for a connection between objects-concepts and definite-indefinite (not in terms of infinite) distinctions. This subsection is just to again call attention as to what do Frege consider arithmetic parameters such as a to be (objects, concepts, or something else...).

3 Properties and objects

3.1 Why numbers are not properties

Frege offers two arguments in support of that premise (from the title). All points in this section were made in parts 21-25 from the original work.

3.1.1 Argument 1

First, he notes the presence of an element of "human choice" He gives an example with the *Iliad*. He claims that were numbers properties, only an individual one should have been assigned to the object. He proceeds to explain that one may be referring to the poem as a whole, its pages, songs, etc. (by using the *Iliad*) From there, according to Frege, it follows that numbers cannot possibly be properties such as colour or solidity because they require an exclusive property. An example will be that an apple is red and cannot be another colour depending on choice made by different individuals.

Comment: This does not seem enough or even correct for me. I find two problems with this reasoning both related to the nature of the argument. First, let us examine the "human choice" element. It is interesting to see whether there exists a single object (for now let's consider spatial objects only) for which there cannot be human choice causing multiple properties of the same kind. Even if there is, it is only natural to assume that an apple can have colour, yet it is no clear if it is red or say yellow because one may refer to either its outside peel or its interior substance. It seems then that if that is a criteria for being a property collection, numbers are not different from colour. Semantically, the argument may be simplified as by agreement that the *Iliad* refers to the poem as a whole, or for the idea of Homer's work, while if we want to infer about the "number property" of its pages, the object of the statements have to be "pages of the *Iliad*". Thus, even if there is any truth in this argument, I think the problem is with the naming of the object rather than the subject (the property).

There is also the problem of requiring objects to possess a single property from a property collection[3]. An example of this would be that something cannot be both red and green. Of course, here we risk falling in the same trap as above in terms of to what we refer by saying some term such as the *Iliad* i.e. are we talking about the colour of the cover, the pages, the ink of the words, etc. It is important to observe that there are definitions, perhaps, under which Frege's requirement is not needed. An example may be how we consider differentiability of a function. It is widely accepted,

[3]Not really sure about any specific terminology here. I am referring to red as a property / property instance and to colour as the property collection, for example.

that this is a property collection of functions, yet more complex functions such as piece-wise, say $f(x) = |x|$, which is differentiable at all real values of x except for $x = 0$. Thus, it is not very clear what is meant by differentiability of a function. One may argue that it is differentiable because we may calculate the derivative of the function, but its domain will be restrained. This, however, is not ideal because it may result in statements like every function is differentiable but the domain of the derivative is the empty set or something similar of no value[4].

This issue is actually an opportunity for a defense of Frege's argument because one may attempt to introduce an additional property instance in a collection called a *multiproperty*. For example, this may refer to something being both red and green, and is thus multicolored. On one hand, this is good news for the object-logicists because it is harder to introduce a concept like "multi-numbered" because it does not have an intuitive accepted meaning, whereas multicolouredness is a somewhat common idea for certain things. On the other, however, there are two obvious problems. First, there is again the problem of definition's usefulness given that multicolored apple does not really give any information about the colours an apple "possesses". Second, there is the problem of artificially creating another property collection. Namely, this is whether some object has a multiproperty for some of its property collections or not.

Somewhat unrelated (at least to *Die Grundlagen*), the last point gives an opportunity for a meta speculations to take place in terms of the very nature of properties and which forms[5] possess them. Namely, one may attempt to construct a somewhat self-reference property or if not just say something like: there exists a property collection that refers to whether a form possesses any other properties, hence it contains two properties - "possesses another property, does not possess another property". This, of course, is meaningless because depending on definition everything will have this property, or no stuff like Frege's concepts will be at risk of having it because of their definitions.

3.1.2 Argument 2

Frege's second argument about why numbers are not properties has to do with what kind of things possess properties. According to Frege, the answer to this are not objects in general, but specifically *spatial* objects such as apples. Given that Frege talks about property collections like solidity and colour as examples. He then goes on to point that numbers can also refer to non-spatial objects. This for him constitutes a big difference.

Comment: The latter part is clearly correct given an obvious example is cardinality of sets which are non-spatial. There is the problem of whether such things even exist in the first place, but given his first principle, I believe this is a clear enough point.

The first part, however, is in my opinion close to being an assumption (about normal property collections such as colour applying only to spatial objects). Again it may be useful, to consider differentiability at a point. Of course, one may argue that this is far beyond arithmetic, and given that functions are typically non-differentiable because of geometric (and thus, according to Frege, possibly synthetic) reasons, it may be out of scope. Then again, it is clear that values (and not

[4] I support Frege's opinion that definitions are judged by their usefulness especially when it comes to *a priori* analytic subjects.

[5] Here I am using the term *form* as basically different stuff like objects, concepts, relations, etc.

points) at which a function is compiled possess that property. We can also propose another examples such as domains of parameters (for example requiring a cardinal number different than 0 because of a division arithmetic operation). This is, however, too close to numbers so it may be a fallacy. Another example may be validity of a logical statement. This is a non-spatial object and validity is a property. This, I think, causes Frege's argument problems.

There are potential grounds for refuting this opposition by saying that validity, soundness, and hence it may be possible to create a one-to-one correspondence by assigning numbers (like a boolean function) to the properties in the collection like saying sound is one and not sound is zero. This may be potential opportunity to reduce this whole system to numbers and thus say it's a fallacy. However, the same can be applied to any "normal" property collections like colours because them referring to spatial objects will mean their cardinality is at worst countable infinity, hence covered by cardinal numbers.